Sample Spaces, Events, Probability

CS 3130/ECE 3530: Probability and Statistics for Engineers

Jan 9, 2025



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Examples:

 $A = \{3, 8, 31\}$ $B = \{\text{apple, pear, orange, grape}\}$ Not a valid set definition: $C = \{1, 2, 3, 4, 2\}$



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The "empty" or "null" set has no elements:

$$\emptyset = \{ \}$$

Integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

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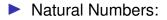
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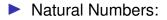


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Real Numbers:

 $\mathbb{R}=$ "any number that can be written in decimal form"

$$5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159 \ldots \in \mathbb{R}$$

Alternate way to define natural numbers:

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Rationals:

$$\mathbb{Q} = \{ p/q : p,q \in \mathbb{Z}, q
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- $A \subseteq A$ for any set A (but $A \not\subset A$)

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- Pick a ball from a bucket of red/black balls: $\Omega = \{R, B\}$
- Tossing 2 coins?
- Shuffling deck of 52 cards?



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- You flip a coin and it comes up "heads": $\{H\} \subseteq \{H, T\}$
- Your code takes longer than 5 seconds to run: $(5,\infty)\subseteq\mathbb{R}$

Set Operations: Union

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Note: If $A \cap B = \emptyset$, we say A and B are **disjoint**.

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Example: $A = \{1,3,5\} \quad \text{``an odd roll''} \\ A^c = \{2,4,6\} \quad \text{``an even roll''}$

Set Operations: Difference

Definition

The **difference** of a set $A \subseteq \Omega$ and a set $B \subseteq \Omega$, denoted A - B, is the set of all elements in Ω that are in A and are not in B.

Example: $A = \{3, 4, 5, 6\}$ $B = \{3, 5\}$ $A - B = \{4, 6\}$

Note: $A - B = A \cap B^c$

DeMorgan's Law

Complement of union or intersection:

$$(A\cup B)^c = A^c \cap B^c$$

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Venn Diagrams



Check whether the following statements are true or false. (Hint: you might use Venn diagrams.)

Probability

Definition

A probability function on a finite sample space Ω assigns every event $A \subseteq \Omega$ a number in [0, 1], such that

1.
$$P(\Omega) = 1$$

2.
$$P(A \cup B) = P(A) + P(B)$$
 when $A \cap B = \emptyset$

P(A) is the **probability** that event A occurs.

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►
$$P({1}) = 1/6$$

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Repeated Experiments

If we do two runs of an experiment with sample space $\Omega,$ then we get a new experiment with sample space

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Properties: Order matters: $(1,2) \neq (2,1)$ Repeats are possible: $(1,1) \in \mathbb{N} \times \mathbb{N}$

Repeating an experiment n times gives the sample space

$$egin{aligned} \Omega^n &= \Omega imes \cdots imes \Omega & (n ext{ times}) \ &= \{(x_1, x_2, \dots, x_n) : x_i \in \Omega ext{ for all } i\} \end{aligned}$$

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 (*n* times)
= {(x₁, x₂, ..., x_n) : x_i $\in \Omega$ for all *i*}

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If $|\Omega| = k$, then $|\Omega^n| = k^n$.

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Union of two overlapping events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number's digits is 5

Permutations

A **permutation** is an ordering of an *n*-tuple. For instance, the *n*-tuple (1, 2, 3) has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$

 $(2, 3, 1), (3, 1, 2), (3, 2, 1)$

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The number of unique orderings of an *n*-tuple is *n* factorial:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2$$

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How many ways can you rearrange (1, 2, 3, 4)?