

Bayesian philosophy, non-informative priors, exchangeability

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Outline

- Bayesian vs. frequentist
- Uninformative priors
- Exchangeability, de Finetti's theorem

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- Uninformative priors
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Bayesian vs. Frequentist

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- Frequentist: θ is some fixed parameter, no randomness
 - We want to estimate it from observations

$$\theta_{ML} = \operatorname{argmax}_{\theta} \sum_{i=1}^N \log p(\mathbf{x}_i | \theta)$$

- How to quantify your uncertainty?
 - confidence level, note that θ_{ML} is a R.V., but θ is not.

Bayesian vs. Frequentist

- Let us consider to estimate a parameter θ , e.g., the chance of head (tossing a coin), from observed data $\mathbf{x}_1, \dots, \mathbf{x}_N$
- Bayesian: θ is a random variable as well!

Bayesian vs. Frequentist

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- Bayesian: θ is a random variable as well!
 - We want to estimate it from observations

$$p(\theta|\mathcal{D}) \propto p(\theta) \prod_{i=1}^N p(\mathbf{x}_i|\theta)$$

- How to quantify your uncertainty?

Posterior distribution! $p(\theta|\mathcal{D})$

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 - ❑ We can make probabilistic statements about θ (mean, variance, quantiles, etc.).

Bayesian vs. Frequentist

- In Bayesian world, every thing is random! (every variable is a random variable)
- Why is random θ is important?
 - We can encode our **beliefs, previous experience and desires** in the prior $p(\theta)$
 - We can make probabilistic statements about θ (mean, variance, quantiles, etc.).
 - We can make Bayesian prediction that integrates all the possible outcomes

$$p(\mathbf{x}^* | \mathbf{x}_1, \dots, \mathbf{x}_N) = \int p(\mathbf{x}^* | \theta) p(\theta | \mathbf{x}_1, \dots, \mathbf{x}_N)$$

Bayesian vs. Frequentist

- Is Bayesian analysis subjective?
 - Not necessary: Bayesian provides a convenient way to incorporate subjective beliefs (important for AI!) But it can also use uninformative priors (this is objective Bayesian!)
 - Frequentist models make assumptions, too!
 - Whether using frequent or Bayesian models, always **check the assumptions you make**

Outline

- Bayesian vs. frequentist
- **Uninformative priors**
- Exchangeability, de Finetti's theorem

Uninformative priors

- In many cases, we have little idea of what form the distribution should take
- Though conjugate priors are computationally nice, objective Bayesians instead prefer priors which *has little influence* on the posterior distribution. Such a prior is called an *uninformative prior*.
- Let the data speak for themselves

Uninformative priors

- What priors do you have immediately in mind?

Uninformative priors

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Uniform distribution!

Uninformative priors

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Uniform distribution!

Now that I do not know which parameter is more likely to be sampled, let us just assume the chances are equal!

Uninformative priors

- Uniform distribution

For finite states: $p(\lambda) = 1/K$

For finite interval: $p(\lambda) = 1/(b - a)$

$$p(\mathbf{x}|\lambda)$$

Uninformative priors

- Uniform distribution

What about unbounded domains? $\lambda \in \mathbb{R}$

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$$p(\lambda) \propto \text{const}$$

Uninformative priors

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What about unbounded domains? $\lambda \in \mathbb{R}$

$$p(\lambda) \propto \text{const}$$

This is an *improper* prior, because normalization diverges

We can still use it as long as the posterior is *proper*

Uninformative priors

- Problem of uniform distribution: *transformation invariance*

$$p(\lambda) \propto \text{const}$$

$$\lambda = \eta^2$$

Uninformative priors

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$$p(\lambda) \propto \text{const}$$

$$\lambda = \eta^2$$

$$p_\eta(\eta) = p_\lambda(\lambda) \left| \frac{d\lambda}{d\eta} \right| = p_\lambda(\eta^2) 2\eta \propto \eta$$

Uninformative priors

- Problem of uniform distribution: *transformation invariance*

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$$\lambda = \eta^2$$

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When we do variable transformations, the prior is no longer uninformative!

Uninformative priors

- Let us take *translation invariance* into account

If the likelihood takes the form

$$p(x|\lambda) = f(x - \lambda)$$

Uninformative priors

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If the likelihood takes the form

$$p(x|\lambda) = f(x - \lambda)$$

λ is *location* parameter, and the density exhibits *shift invariance*

$$\hat{x} = x + c \quad \hat{\lambda} = \lambda + c$$

$$p(\hat{x}|\hat{\lambda}) = f(\hat{x} - \hat{\lambda})$$

$$p(\mathbf{x}|\lambda)$$

Uninformative priors

- We want to construct a prior that reflects this shift invariance (why: more consist with the likelihood, less influence on the posterior!)

$$p(\mathbf{x}|\lambda)$$

Uninformative priors

- We want to construct a prior that reflects this shift invariance (why: **more consist with the likelihood, less influence on the posterior!**)
- How? We choose a prior that assigns equal probability mass to an arbitrary interval $[A, B]$ as to the shifted interval $[A+c, B+c]$

Uninformative priors

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- How? We choose a prior that assigns equal probability mass to an arbitrary interval $[A, B]$ as to the shifted interval $[A+c, B+c]$

$$\int_A^B p(\lambda) d\lambda = \int_{A+c}^{B+c} p(\lambda) d\lambda$$

Uninformative priors

$$\int_A^B p(\lambda) d\lambda = \int_{A+c}^{B+c} p(\lambda) d\lambda = \int_A^B p(\lambda + c) d\lambda$$

$\lambda = \bar{\lambda} + c$

$$p(\lambda) = p(\lambda + c)$$

$$p(\lambda) \propto \text{const}$$

Uninformative priors

Example: for a Gaussian likelihood

$$p(x|\mu) = \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2} \underline{(x - \mu)^2}\right)$$

Uninformative priors

Example: for a Gaussian likelihood

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shift invariance density

Conjugate prior

$$p(\mu|\alpha, v^2) = \mathcal{N}(\mu|\alpha, v^2) = \frac{1}{\sqrt{2\pi}v} \exp\left(-\frac{1}{2v^2}(\mu - \alpha)^2\right)$$

Uninformative priors

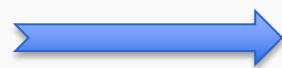
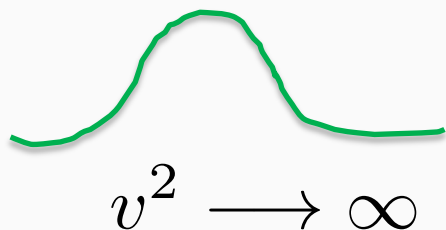
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$p(\mu) \propto \text{const}$

Uninformative priors

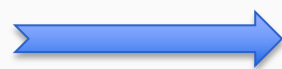
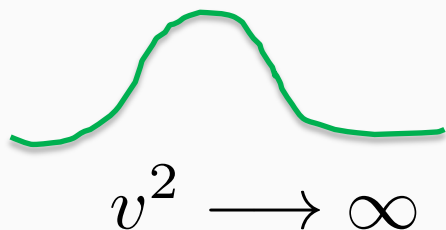
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$$p(\mu) \propto \text{const}$$

Limit of the conjugate prior

Uninformative priors

- Let us take *translation invariance* into account

If the likelihood takes the form

$$p(x|\sigma) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right) \quad \sigma > 0 \quad f \text{ normalizes regularly}$$

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$$p(x|\sigma) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right) \quad \sigma > 0 \quad f \text{ normalizes regularly}$$

σ is *scale* parameter, and the density exhibits *scale invariance*

$$\hat{x} = cx \quad \hat{\sigma} = c\sigma$$

$$p(\hat{x}|\hat{\sigma}) = \frac{1}{\hat{\sigma}} f\left(\frac{\hat{x}}{\hat{\sigma}}\right) \quad \text{Verify it by yourself}$$

Uninformative priors

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Uninformative priors

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- How, consider an arbitrary interval $[A, B]$, the prior should assign equal mass over an arbitrary scaled interval $[A/c, B/c]$

$$\int_A^B p(\sigma) d\sigma = \int_{A/c}^{B/c} p(\sigma) d\sigma$$

Uninformative priors

$$\bar{\sigma} = c\sigma$$

$$\int_A^B p(\sigma) d\sigma = \int_{A/c}^{B/c} p(\sigma) d\sigma = \int_A^B p\left(\frac{1}{c}\sigma\right) \frac{1}{c} d\sigma$$

$$p(\sigma) = p\left(\frac{1}{c}\sigma\right) \frac{1}{c}$$

$$p(\sigma) \propto 1/\sigma$$

Uninformative priors

Example: for a Gaussian likelihood

$$p(x|\sigma) = \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \exp\left(-\frac{1}{2} \left[\frac{x - \mu}{\sigma}\right]^2\right)$$

Uninformative prior

$$p(\sigma) \propto 1/\sigma \quad \xrightarrow{\lambda = 1/\sigma^2} \quad p(\lambda) \propto 1/\lambda$$

Conjugate prior

$$p(\lambda|a, b) = \text{Gam}(\lambda|a, b) \propto \lambda^{a-1} \exp(-b\lambda)$$

$$a = 0, b = 0 \quad \xrightarrow{\quad} \quad p(\lambda) \propto 1/\lambda$$

Uninformative Priors

- Jeffreys priors

$$\pi_J(\boldsymbol{\theta}) \propto |I(\boldsymbol{\theta})|^{1/2}$$

Fisher information

$$I(\theta) = -\mathbb{E}_\theta \left[\frac{d^2 \log p(X|\theta)}{d\theta^2} \right]$$

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Expectation w.r.t $p(X|\theta)$



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Expectation w.r.t $p(X|\theta)$



Note, for vector case, it becomes the Hessian

Jeffreys priors - example

Binomial likelihood

$$X \sim \text{Bin}(n, \theta), 0 \leq \theta \leq 1$$

$$p(x|\theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$$

Jeffreys priors - example

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Let's construct a Jeffreys prior over θ

Jeffreys priors - example

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Let's construct a Jeffreys prior over θ

$$\log p(x|\theta) = x \log \theta + (n - x) \log(1 - \theta)$$

$$\frac{d}{d\theta} \log p(x|\theta) = \frac{x}{\theta} - \frac{n - x}{1 - \theta}$$

$$\frac{d^2}{d\theta^2} \log p(x|\theta) = -\frac{x}{\theta^2} - \frac{n - x}{(1 - \theta)^2}$$

Jeffreys priors - example

$$\frac{d^2}{d\theta^2} \log p(x|\theta) = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2}$$



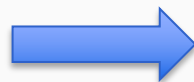
$$\mathbb{E}[x] = n\theta$$

$$I(\theta) = -\mathbb{E}_\theta \left[\frac{d^2 \log p(x|\theta)}{d\theta^2} \right]$$

$$= \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$= \frac{n}{\theta(1-\theta)}$$



$$\pi_J(\theta) = I(\theta)^{\frac{1}{2}} \propto \theta^{-\frac{1}{2}} (1-\theta)^{-\frac{1}{2}}$$

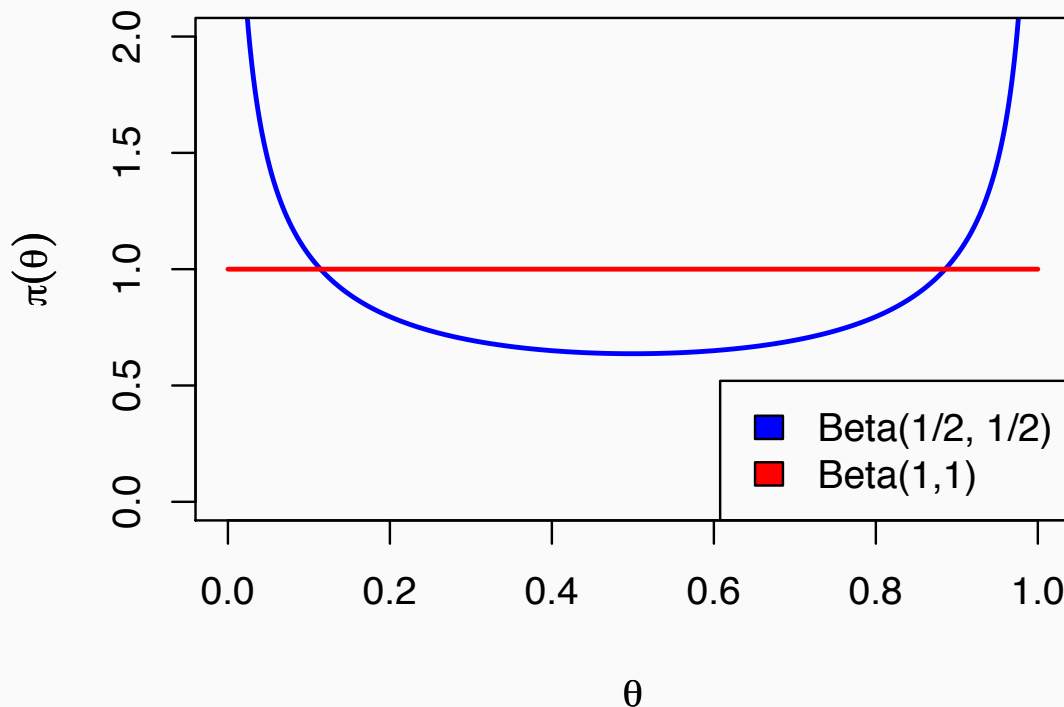
Beta($\frac{1}{2}$, $\frac{1}{2}$)

Jeffreys priors - example

Binomial likelihood

$$X \sim \text{Bin}(n, \theta), 0 \leq \theta \leq 1$$

$$p(x|\theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$$



Data takes least effect

$$\theta = \frac{1}{2}$$

Data takes greatest effect

$$\theta = 0 \text{ or } 1$$

Prior is consistent with the data effect!

Jeffreys priors – translation invariance

- Let us consider a general translation $\phi = h(\theta)$

What is the Jeffreys prior over ϕ ?

$$\pi_J(\phi) \propto |\mathbf{I}(\phi)|^{\frac{1}{2}}$$

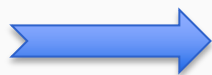
Use Chain rule

$$\begin{aligned}\mathbf{I}(\phi) &= -\mathbb{E} \left[\frac{d^2 \log p(X|\phi)}{d\phi^2} \right] \\ &= -\mathbb{E} \left[\frac{d^2 \log p(X|\theta)}{d\theta^2} \left(\frac{d\theta}{d\phi} \right)^2 + \frac{d \log p(X|\theta)}{d\theta} \frac{d^2 \theta}{d\phi^2} \right]\end{aligned}$$

Jeffreys priors – translation invariance

We know $\mathbb{E} \left[\frac{d \log p(X|\theta)}{d\theta} \right] = 0$ Why?

$$\forall \theta, \int p(X|\theta) dX = 1$$



$$\begin{aligned} 0 &= \frac{d}{d\theta} \int p(X|\theta) dX \\ &= \int \frac{dp(X|\theta)}{d\theta} \frac{p(X|\theta)}{p(X|\theta)} dX \\ &= \int \left[\frac{dp(X|\theta)}{d\theta} \frac{1}{p(X|\theta)} \right] p(X|\theta) dX \\ &= \int \left[\frac{d \log p(X|\theta)}{d\theta} \right] p(X|\theta) dX \\ &= \mathbb{E} \left[\frac{d \log p(X|\theta)}{d\theta} \right] \end{aligned}$$

Jeffreys priors – translation invariance

$$\mathbf{I}(\phi) = -\mathbb{E} \left[\underbrace{\frac{d^2 \log p(X|\theta)}{d\theta^2} \left(\frac{d\theta}{d\phi} \right)^2}_{\mathbf{I}(\theta)} + \underbrace{\frac{d \log p(X|\theta)}{d\theta} \frac{d^2\theta}{d\phi^2}}_0 \right]$$



$$\mathbf{I}(\phi) = \mathbf{I}(\theta) \left(\frac{d\theta}{d\phi} \right)^2$$



$$\sqrt{\mathbf{I}(\phi)} = \sqrt{\mathbf{I}(\theta)} \left| \frac{d\theta}{d\phi} \right|$$

Jeffreys priors – translation invariance

$$\sqrt{\mathbf{I}(\phi)} = \sqrt{\mathbf{I}(\theta)} \left| \frac{d\theta}{d\phi} \right|$$

Now, we can see

When we directly construct Jeffreys prior

$$\pi_J(\phi) \propto \sqrt{\mathbf{I}(\phi)} \longrightarrow \text{The same!}$$

When we derive the prior via variable transformation

$$\pi_J(\phi) \propto \sqrt{\mathbf{I}(h^{-1}(\phi))} \left| \frac{d\theta}{d\phi} \right| = \sqrt{\mathbf{I}(\theta)} \left| \frac{d\theta}{d\phi} \right|$$

Jeffreys priors – translation invariance

- Now we can show, for a Gaussian likelihood

$$p(x|\mu, \sigma) = \mathcal{N}(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right)$$

$$\pi_J(\mu) \propto 1$$

$$\pi_J(\sigma) \propto \frac{1}{\sigma}$$

Leave it as your
exercise

Jeffreys prior

- Usually not conjugate
 - If you choose Jeffreys prior over μ, σ for a Gaussian likelihood

The posterior of μ will be a student t distribution
- Works well for single parameter, but not for models with multidimensional parameters (e.g., poor convergence properties, not very reasonable estimates)

Reference priors

- formalize what exactly we mean by an “uninformative prior”: *a function that maximizes some measure of distance or divergence between the posterior and prior, as data observations are made.*
- A commonly used divergence is KL divergence

$$\text{KL}(p(\theta|t)||p(\theta)) = \int p(\theta|t) \log \frac{p(\theta|t)}{p(\theta)} d\theta$$

Reference priors

- We choose the prior that maximizes the expected KL divergence between the posterior and the prior

$$\begin{aligned} I(\Theta, T) &= \int p(t) \int p(\theta|t) \log \frac{p(\theta|t)}{p(\theta)} d\theta dt \\ &= \int \int p(\theta, t) \log \frac{p(\theta, t)}{p(\theta)p(t)} d\theta dt \end{aligned}$$



$$p^*(\theta) = \arg \max_{p(\theta)} I(\Theta, T)$$

Mutual information

Outline

- Bayesian vs. frequentist
- Uninformative priors
- Exchangeability, de Finetti's theorem

Bayesian vs. Frequentist

- Given a distribution $p(x|\theta)$ governed by θ
- Frequentist: I believe θ is objective constant, I need to estimate it from IID samples x_1, \dots, x_N
- Bayesian: I believe θ is some *latent random variable* – it was first sampled from a *prior distribution* $p(\theta)$, then given θ , we sample the observations x_1, \dots, x_N

Bayesian vs. Frequentist

- Bayesian: I believe θ is some *latent random variable* – it was first sampled from a *prior distribution* $p(\theta)$, then given θ , we sample the observations x_1, \dots, x_N
- Although it sounds a philosophical choice, can we justify Bayesian modeling with some mathematical evidence?

Exchangeability

- Most statistical analysis are based on IID observations x_1, \dots, x_N

$$p(X_1 = x_1, \dots, X_N = x_N) = \prod_{n=1}^N p(X_n = x_n)$$

- While the assumption is convenient, it may not be reasonable in many problems: weather conditions, stock prices, precipitation, disease rate, ...
- Exchangeability is a much *weaker* assumption

Exchangeability

- Finite exchangeability: Given N random variables, and arbitrary permutation $\pi(1), \dots, \pi(N)$

$$X_1, \dots, X_N \stackrel{d}{=} X_{\pi(1)}, \dots, X_{\pi(N)}$$



$\forall x_1, \dots, x_N$ in the domain

$$p(X_1 = x_1, \dots, X_N = x_N) = p(X_1 = x_{\pi(1)}, \dots, X_N = x_{\pi(N)})$$

e.g. $p(X_1 = 1, X_2 = 2, X_3 = 3) = p(X_1 = 2, X_2 = 3, X_3 = 1)$
 $= p(X_1 = 3, X_2 = 1, X_3 = 2) = \dots$

Exchangeability – infinite sequence

- An infinite sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is exchangeable if $\forall n = 1, 2, \dots$

$$X_1, \dots, X_n \stackrel{d}{=} X_{\pi(1)}, \dots, X_{\pi(n)}, \quad \forall \pi \in S(n),$$

where $S(n)$ are all possible permutations over the first n variables

Exchangeability

- Essentially assume the *symmetry* of the density

$$p(X_1 = x_1, \dots, X_N = x_N) = p(X_1 = x_{\pi(1)}, \dots, X_N = x_{\pi(N)})$$



Exchangeability - one specific example

- Polya's Urn

- Given an urn with B_0 black and W_0 white balls, draw balls with the following procedure

- (1) Draw a ball at random from the urn and note its color
 - (2) If the ball is black then $X_i = 1$; otherwise $X_i = 0$
 - (3) $i = i + 1$
 - (4) Place α balls of the same color in the urn
 - (5) Goto (1)

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$$p(1, 1, 0, 1) = \frac{B_0}{B_0 + W_0} \times \frac{B_0 + a - 1}{B_0 + W_0 + a - 1} \times \frac{W_0}{B_0 + W_0 + 2a - 2} \times \frac{B_0 + 2a - 2}{B_0 + W_0 + 3a - 3}$$

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The sequence $\{X_i, i \geq 1\}$ is exchangeable but not IID

De Finetti's theorem

(de Finetti 1931) A binary sequence $\{X_i\}_{i=1}^{\infty}$ is *exchangeable* iff there exists a *distribution function* F on $[0, 1]$ such that for all n ,

$$p(x_1, \dots, x_n) = \int_0^1 \theta^{t_n} (1 - \theta)^{n - t_n} dF(\theta),$$

where $p(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$ and $t_n = \sum_{i=1}^n x_i$.

$p(\theta)d\theta$

1. There is a latent random variable θ
2. It has a prior distribution

De Finetti's theorem

It further holds that F is the distribution function of the limiting frequency:

$$Y = \bar{X}_\infty = \lim_{n \rightarrow \infty} \sum_i X_i/n, \quad P(Y \leq y) = F(y)$$

and the Bernoulli distribution is obtained by conditioning with $Y = \theta$:

$$P(X_1 = x_1, \dots, X_n = x_n \mid Y = \theta) = \theta^{t_n} (1 - \theta)^{n - t_n}.$$

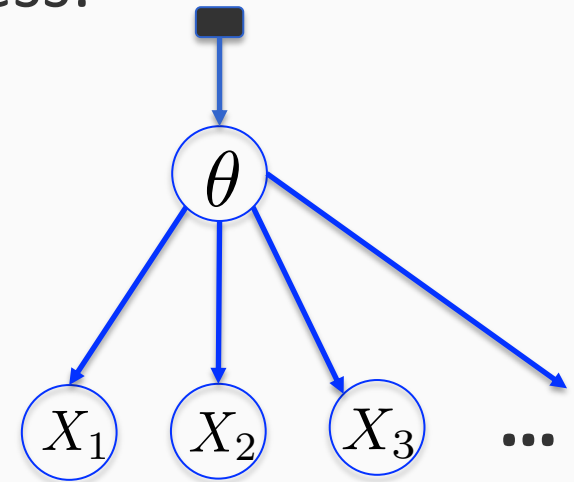
De Finetti's theorem – the underlying sampling process

- If our binary observations $\{X_i\}_{i=1}^{\infty}$ are exchangeable, it implies a hierarchical sampling process:

$$\theta \sim p(\theta)$$

Conditional independent

$$X_1, X_2, \dots | \theta \sim \prod_{i=1}^{\infty} p(X_i | \theta)$$



This justifies Bayesian modeling --- prior distribution objectively exists!

Exchangeability

- Very widely used assumption in Bayesian modeling
- More flexible than IID, but is also restrictive
- Some classical/popular models

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What you need to know

- Bayesian vs. Frequentist
- What is uninformative prior
- What are shift invariance, scale invariance in likelihood? How to derive the corresponding uninformative prior?
- What is Jeffery's prior? Arbitrary translation invariance
- Exchangeability
- De-Finette theorem (how does it justify Bayesian)s