Support Vector Machines: Training with Stochastic Gradient Descent

Machine Learning
Spring 2018

THE UNIVERSITY OF UTAH

The slides are mainly from Vivek Srikumar

Support vector machines

- Training by maximizing margin
- The SVM objective
- Solving the SVM optimization problem
- Support vectors, duals and kernels

SVM objective function

$$\min_{\mathbf{w},b} \quad \frac{1}{2}\mathbf{w}^\top\mathbf{w} + C\sum_{i} \max\left(0, 1 - y_i(\mathbf{w}^\top\mathbf{x}_i + b)\right)$$

Regularization term:
- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

Empirical Loss:
- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

A hyper-parameter that controls the tradeoff between a large margin and a small hinge-loss

Outline: Training SVM by optimization

1. Review of convex functions and gradient descent
2. Stochastic gradient descent
3. Gradient descent vs stochastic gradient descent
4. Sub-derivatives of the hinge loss
5. Stochastic sub-gradient descent for SVM
6. Comparison to perceptron

Outline: Training SVM by optimization

1. **Review of convex functions and gradient descent**
2. Stochastic gradient descent
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Solving the SVM optimization problem

$$\min_{\mathbf{w},b} \quad \frac{1}{2}\mathbf{w}^\top \mathbf{w} + C \sum_i \max\left(0, 1 - y_i(\mathbf{w}^\top \mathbf{x}_i + b)\right)$$

This function is convex in **w,** b

For convenience, use simplified notation:

$\mathbf{w_0} \leftarrow \mathbf{w}$

$\mathbf{w} \leftarrow [\mathbf{w_0}, b]$

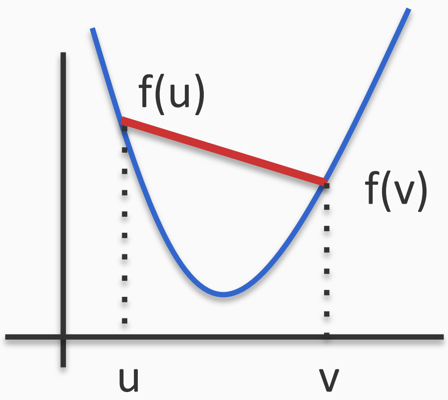
$\boldsymbol{x}_i \leftarrow [\boldsymbol{x}_i, 1]$

$$\min_{\mathbf{w}} \quad \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C \sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

Recall: Convex functions

A function f is convex if for every $\boldsymbol{u}, \boldsymbol{v}$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{v}) \leq \lambda f(\boldsymbol{u}) + (1-\lambda) f(\boldsymbol{v})$$



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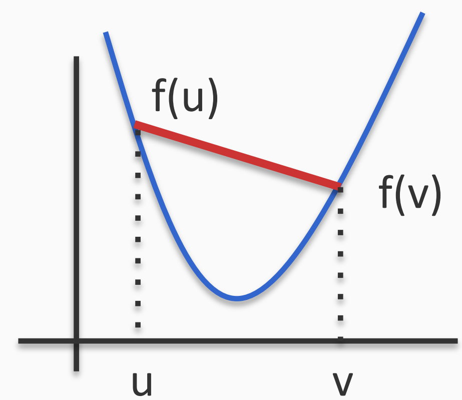
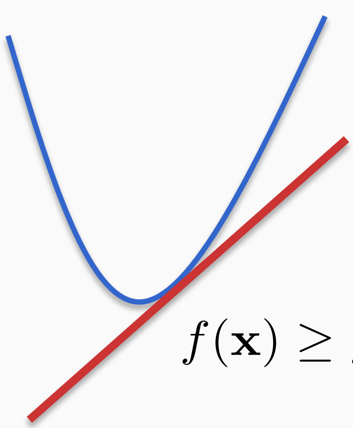
From geometric perspective



Every tangent plane lies below the function

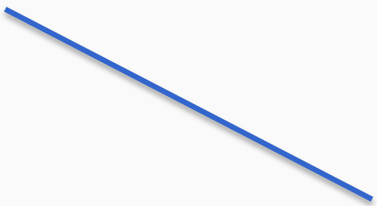


$$f(\mathbf{x}) \geq f(\mathbf{u}) + \nabla f(u)^\top (\mathbf{x} - \mathbf{u})$$

Convex functions

$$f(x) = -x$$

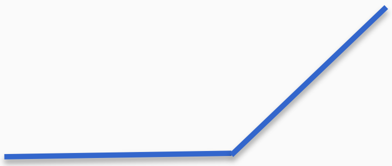
Linear functions



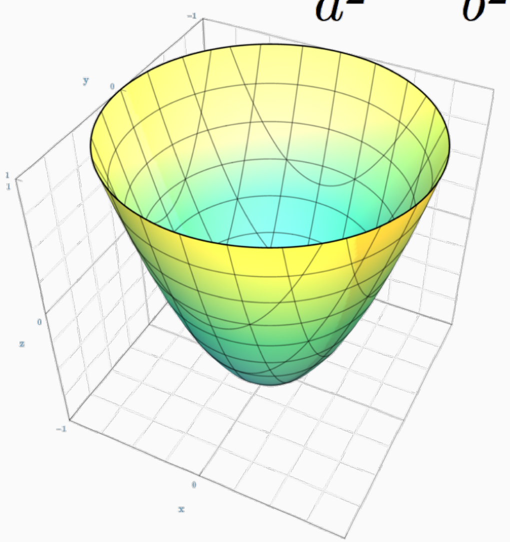
$$f(x) = x^2$$

$$f(x) = \max(0, x)$$

max is convex



$$f(x_1, x_2) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$$



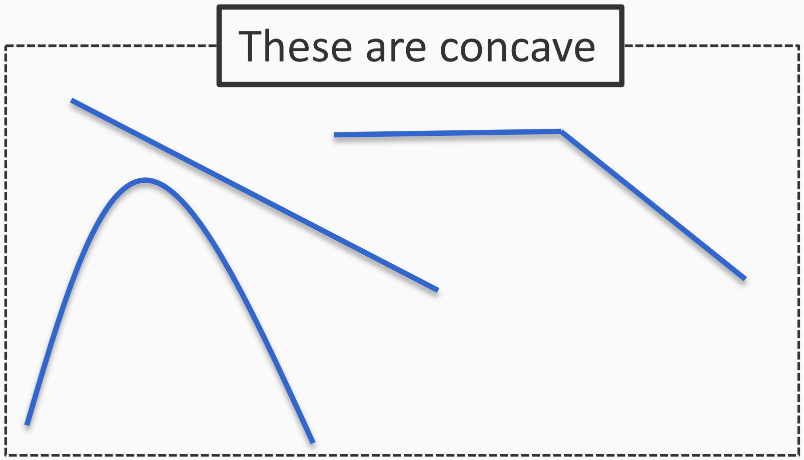
Some ways to show that a function is convex:

1. Using the definition of convexity

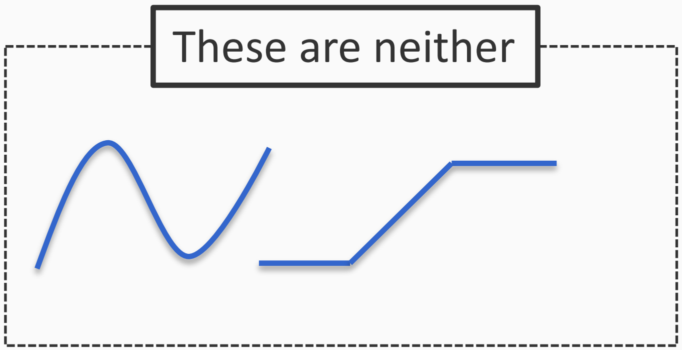
2. Showing that the second derivative is nonnegative (for one dimensional functions)

3. Showing that the second derivative is positive semi-definite (for vector functions)

Not all functions are convex



$$f(\lambda \boldsymbol{u} + (1-\lambda)\boldsymbol{v}) \geq \lambda f(\boldsymbol{u}) + (1-\lambda) f(\boldsymbol{v})$$



Convex functions are convenient

A function f is convex if for every $\boldsymbol{u}, \boldsymbol{v}$ in the domain, and for every $\lambda \in [0,1]$ we have

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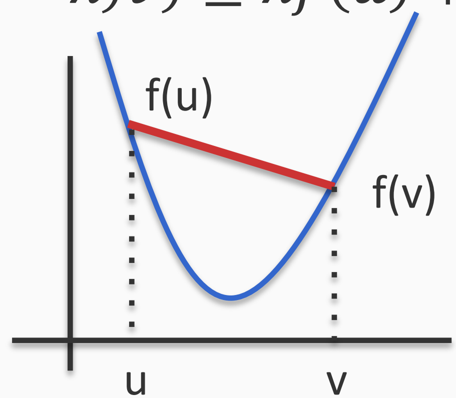


In general: Necessary condition for x to be a minimum for the function f is $\nabla f(x) = 0$

For convex functions, this is both necessary *and* sufficient

Solving the SVM optimization problem

$$\min_{\mathbf{w}} \quad \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C\sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

This function is convex in **w**

- This is a quadratic optimization problem because the objective is quadratic
- Older methods: Used techniques from Quadratic Programming
  - Very slow
- No constraints, can use *gradient descent*
  - Still very slow!

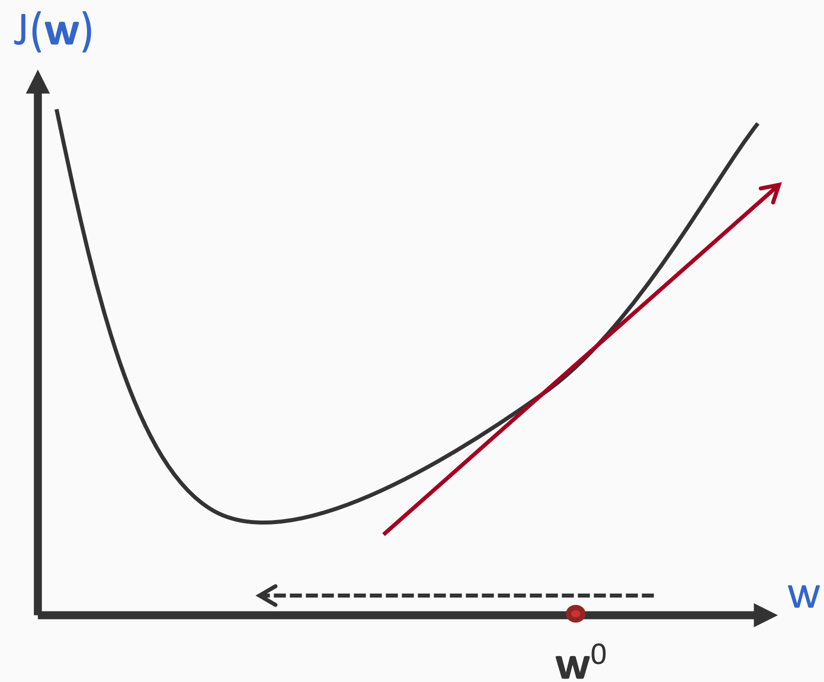
Gradient descent

We are trying to minimize

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C\sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

General strategy for minimizing a function J(**w**)

- Start with an initial guess for **w**, say $\mathbf{w}^0$
- Iterate till convergence:
  - Compute the gradient of J at $\mathbf{w}^t$
  - Update $\mathbf{w}^t$ to get $\mathbf{w}^{t+1}$ by taking a step in the opposite direction of the gradient



Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction

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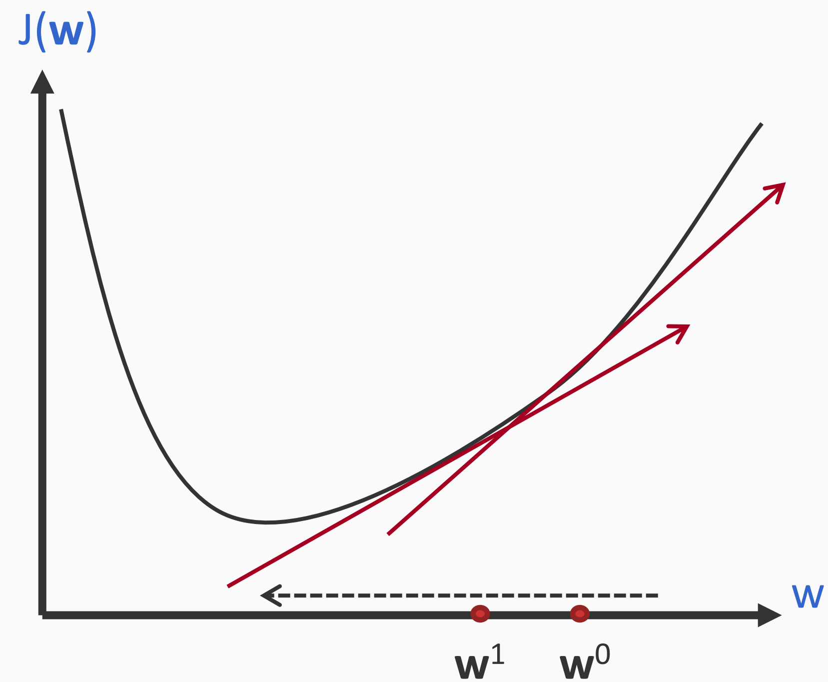

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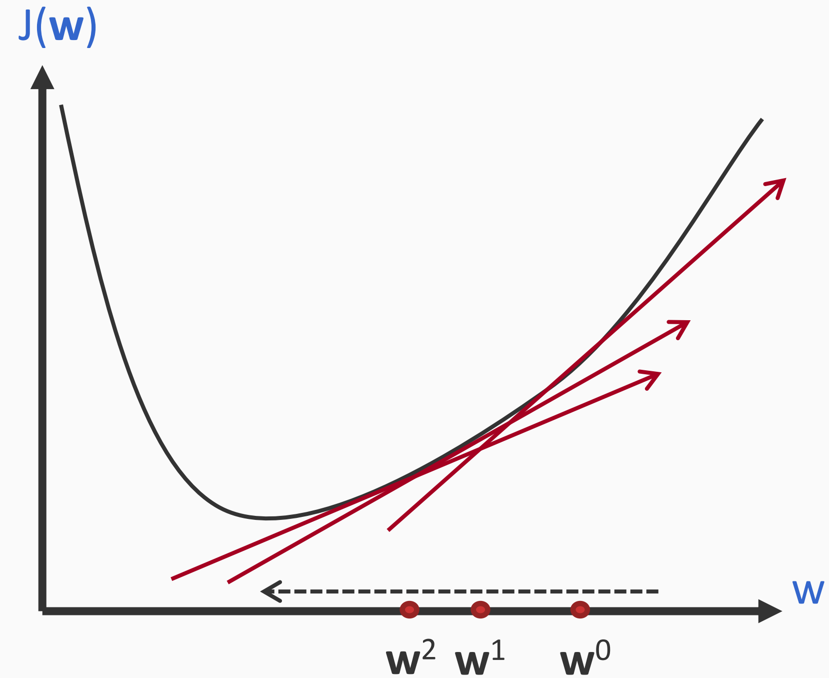

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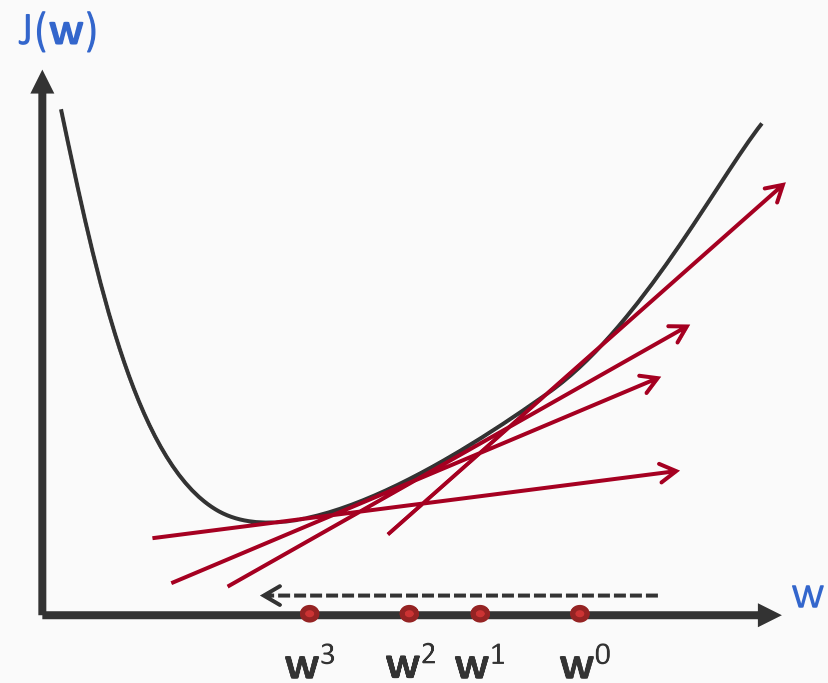

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Gradient descent for SVM

We are trying to minimize

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C\sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

1. Initialize $\mathbf{w}^0$
2. For t = 0, 1, 2, ….
   1. Compute gradient of J(**w**) at $\mathbf{w}^t$. Call it $\nabla J(\mathbf{w}^t)$
   2. Update w as follows:

$$\mathbf{w}^{t+1} = \mathbf{w}^t - r\nabla J(\mathbf{w}^t)$$

r: Called the learning rate .

Outline: Training SVM by optimization

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Gradient of the SVM objective requires summing over the entire training set

Slow, does not really scale

r: Called the learning rate

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C\sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

Stochastic gradient descent for SVM

Given a training set S = $\{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:

3. Return final **w**

$$J(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C \sum_i \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

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$$J^t(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C \cdot N \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

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Number of training examples

3. Return final **w**

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    3. Update: $\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \gamma_t \nabla J^t(\mathbf{w}^{t-1})$
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This algorithm is guaranteed to converge to the minimum of J if γ_t is small enough.

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✓ Stochastic gradient descent

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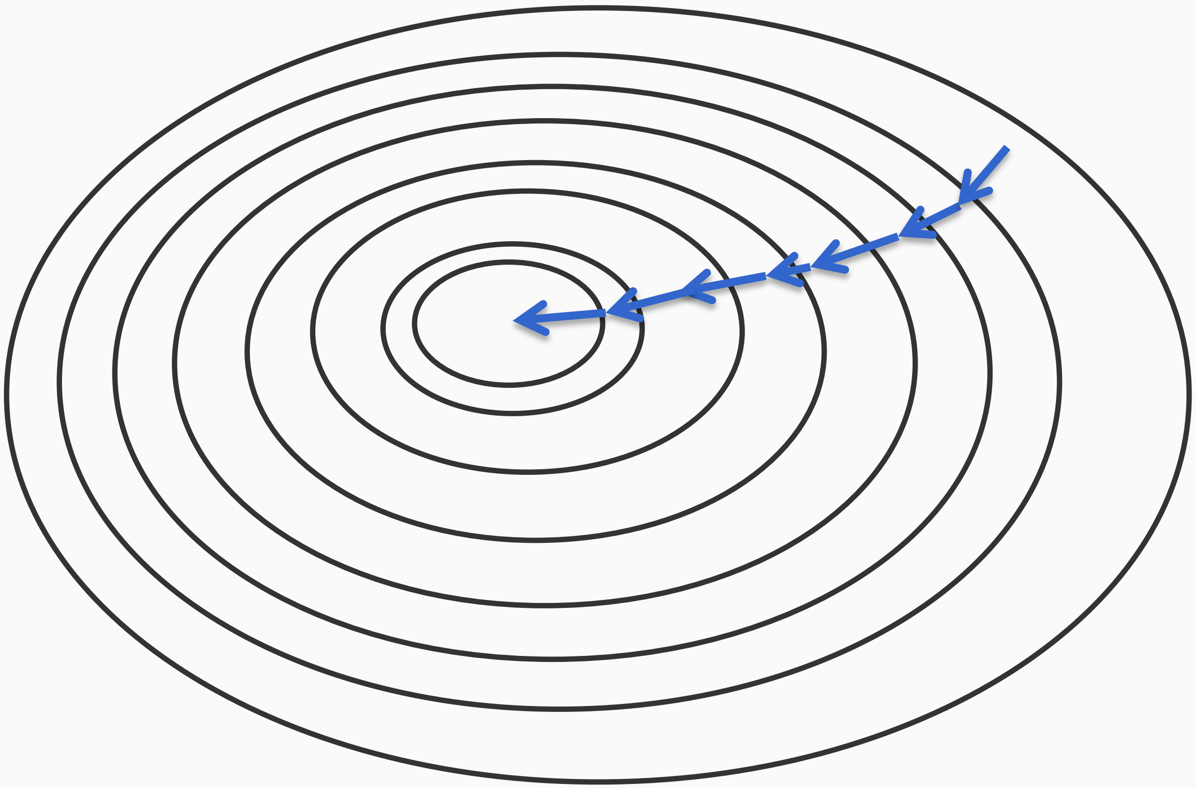
Gradient Descent vs SGD



Gradient descent

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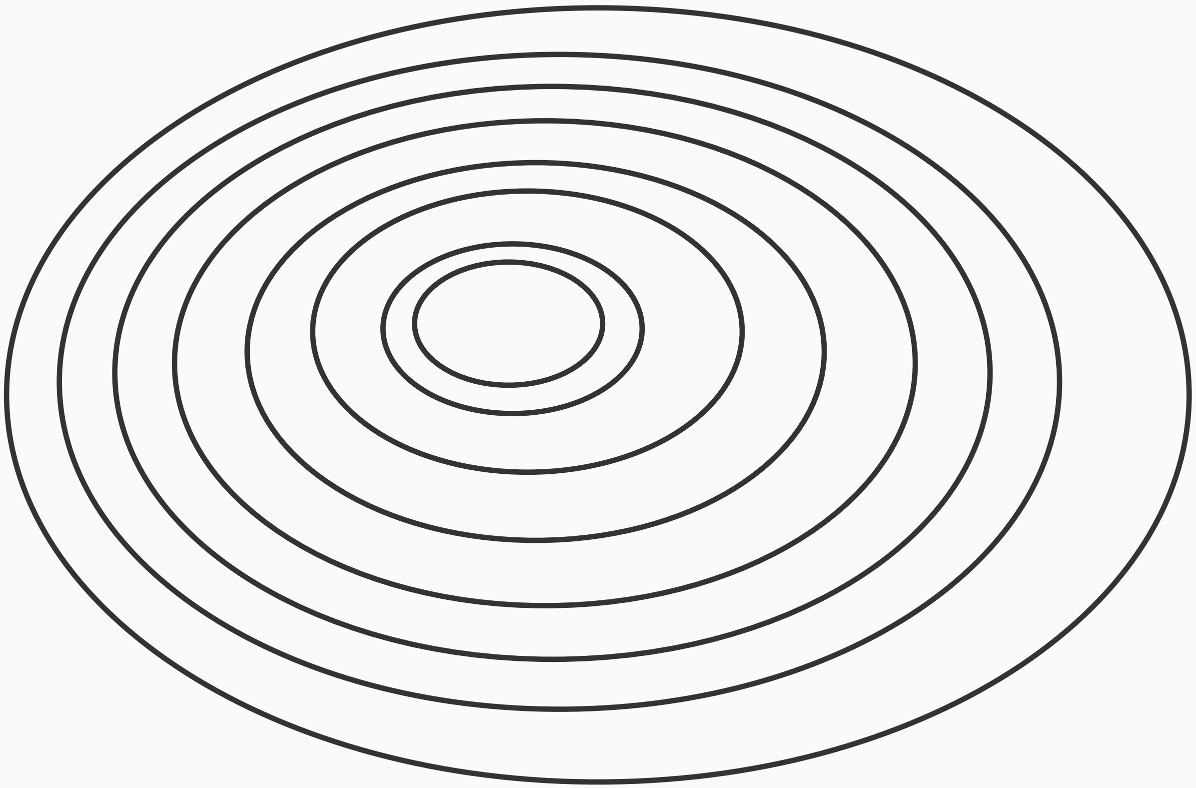

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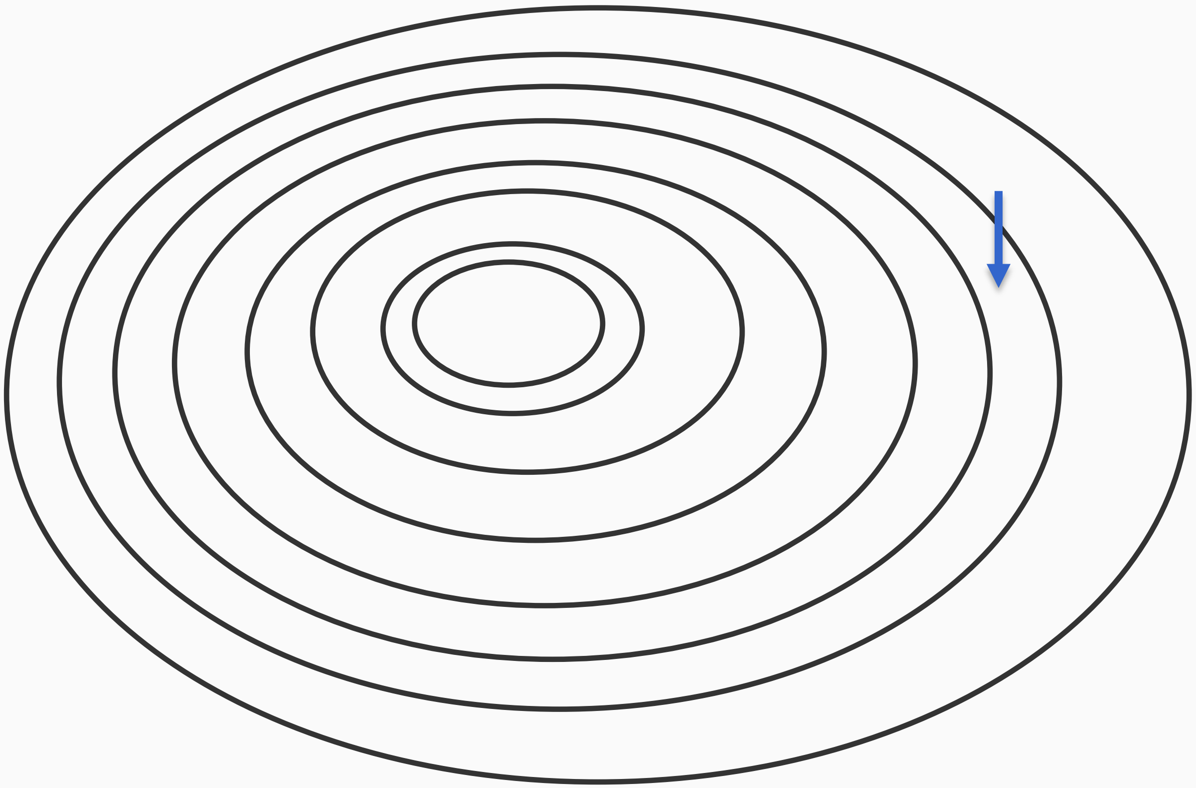

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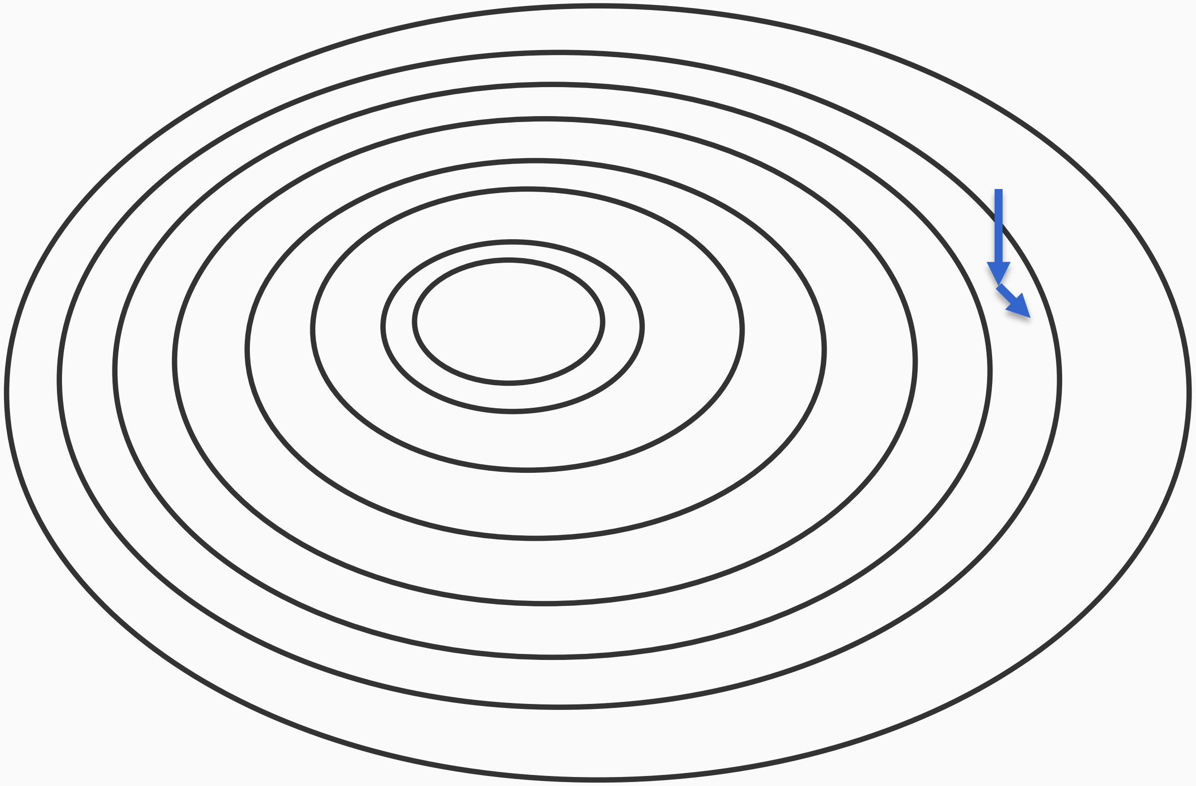

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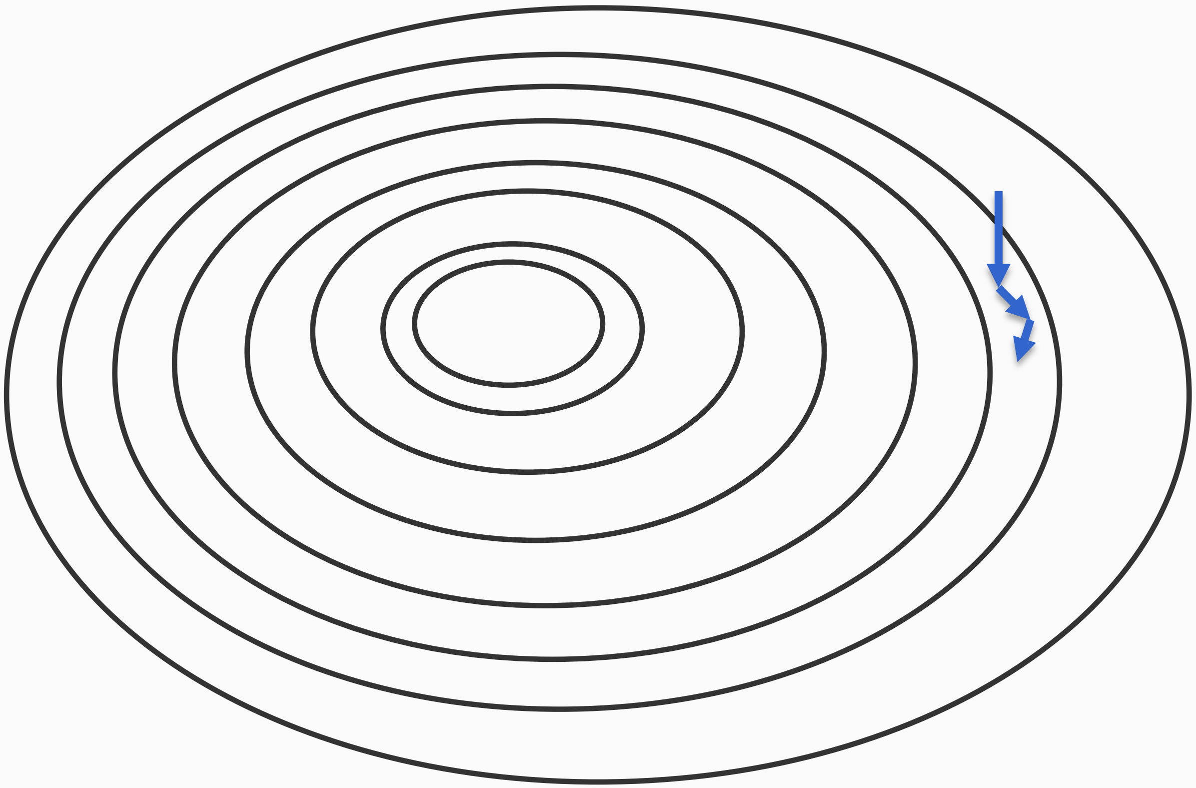

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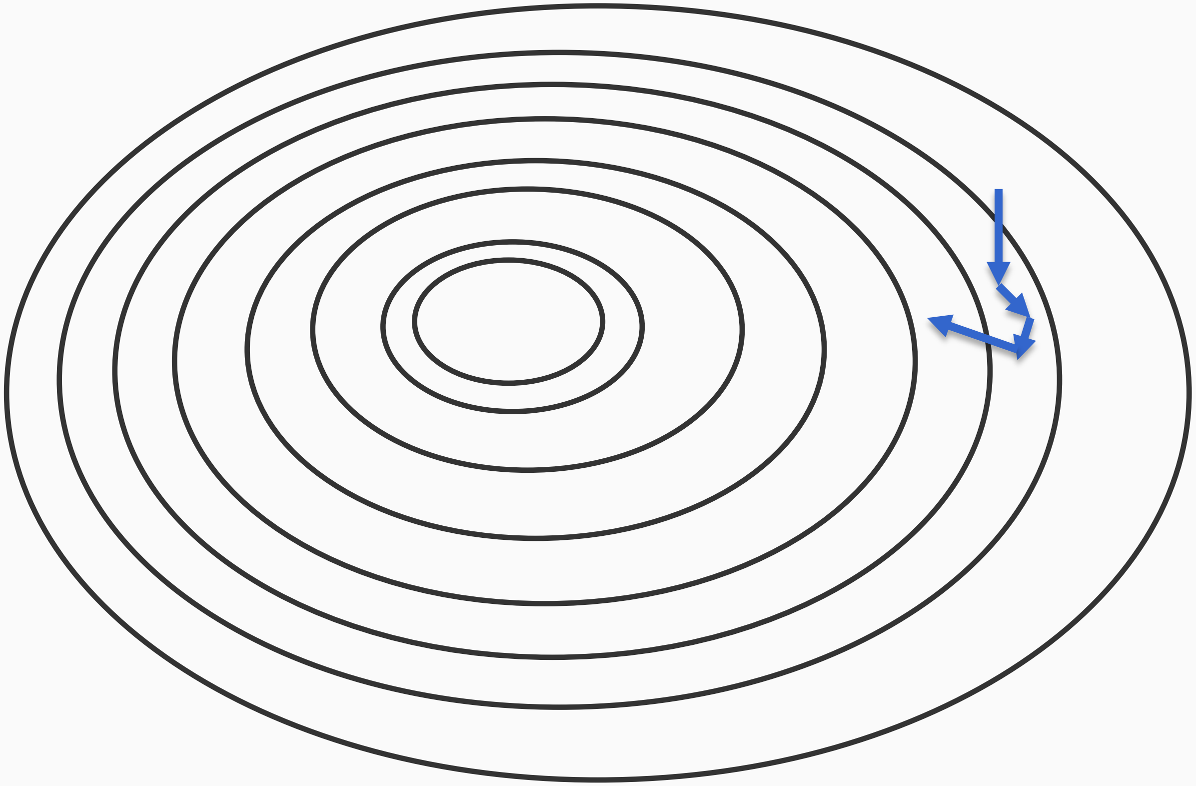

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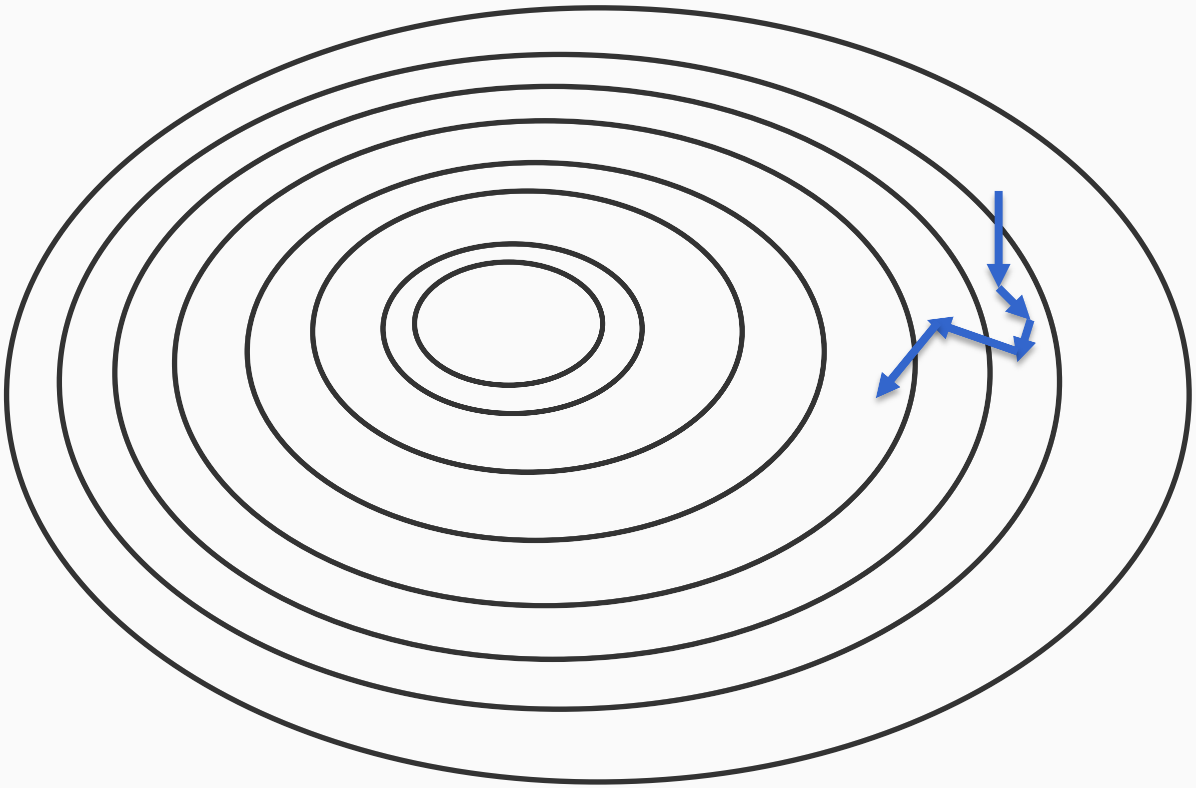

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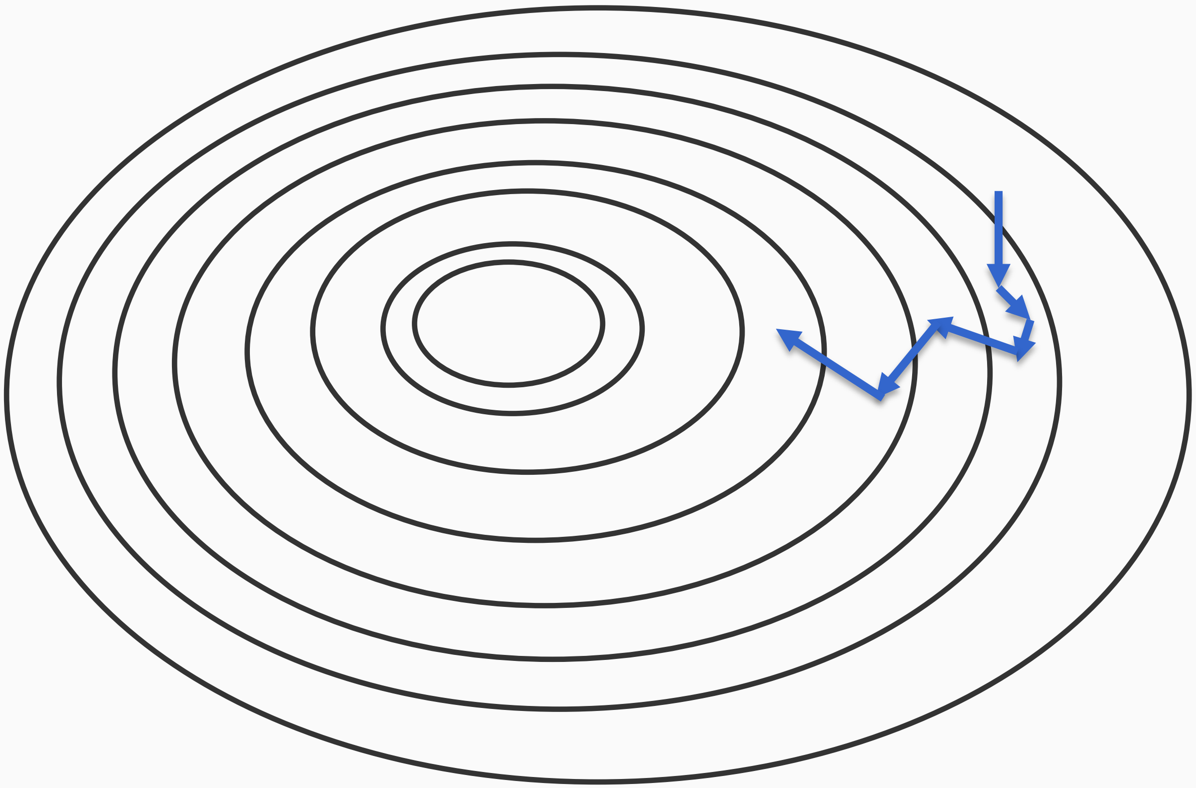

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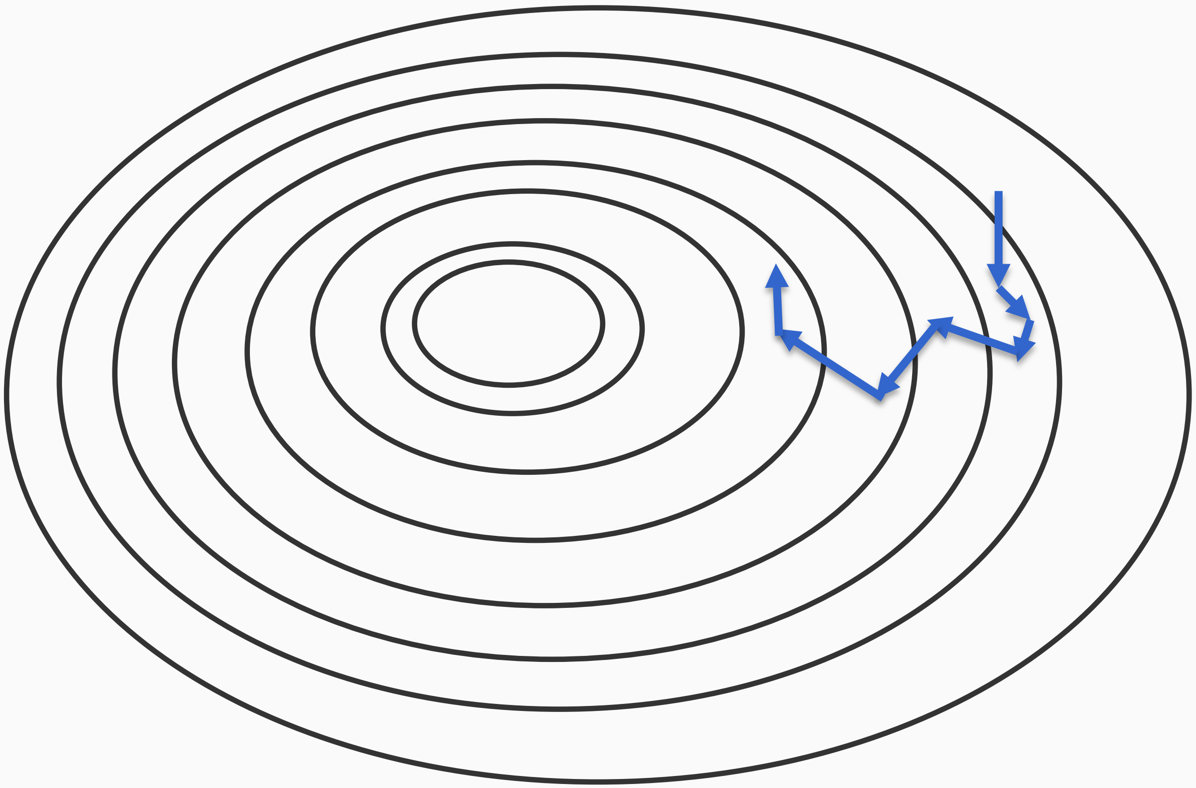

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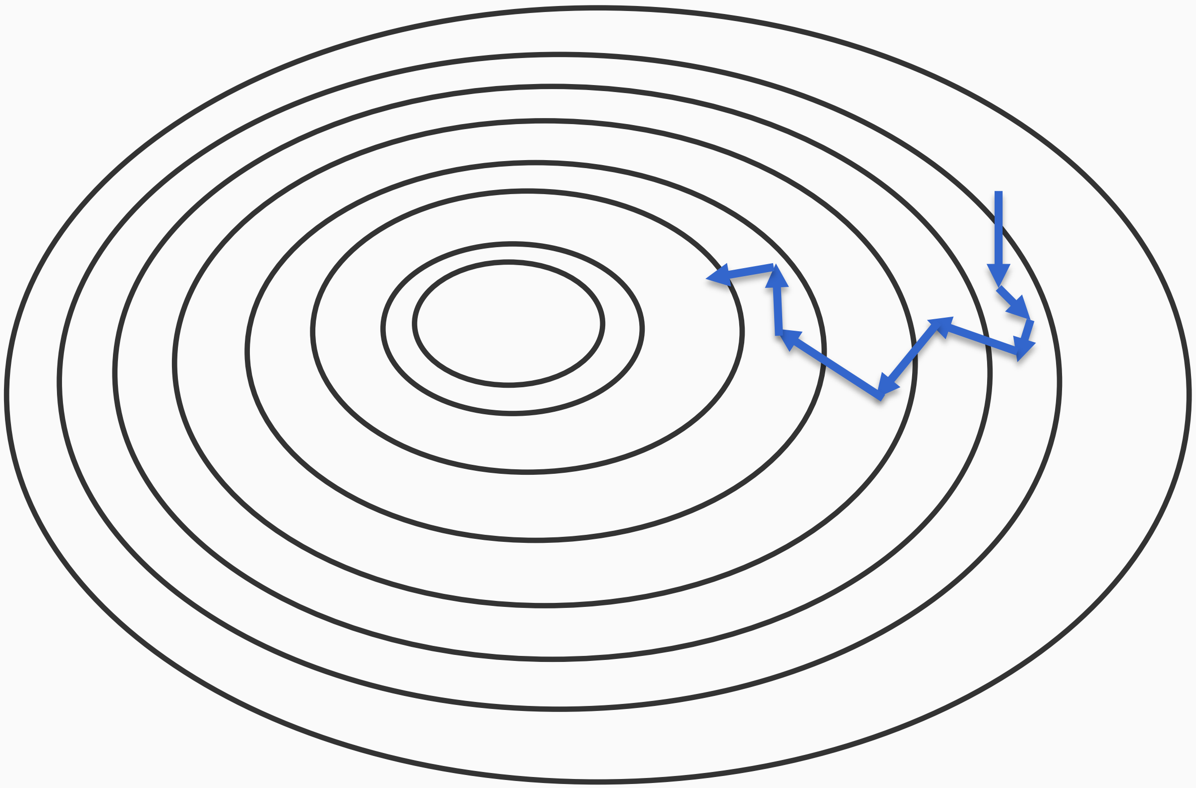

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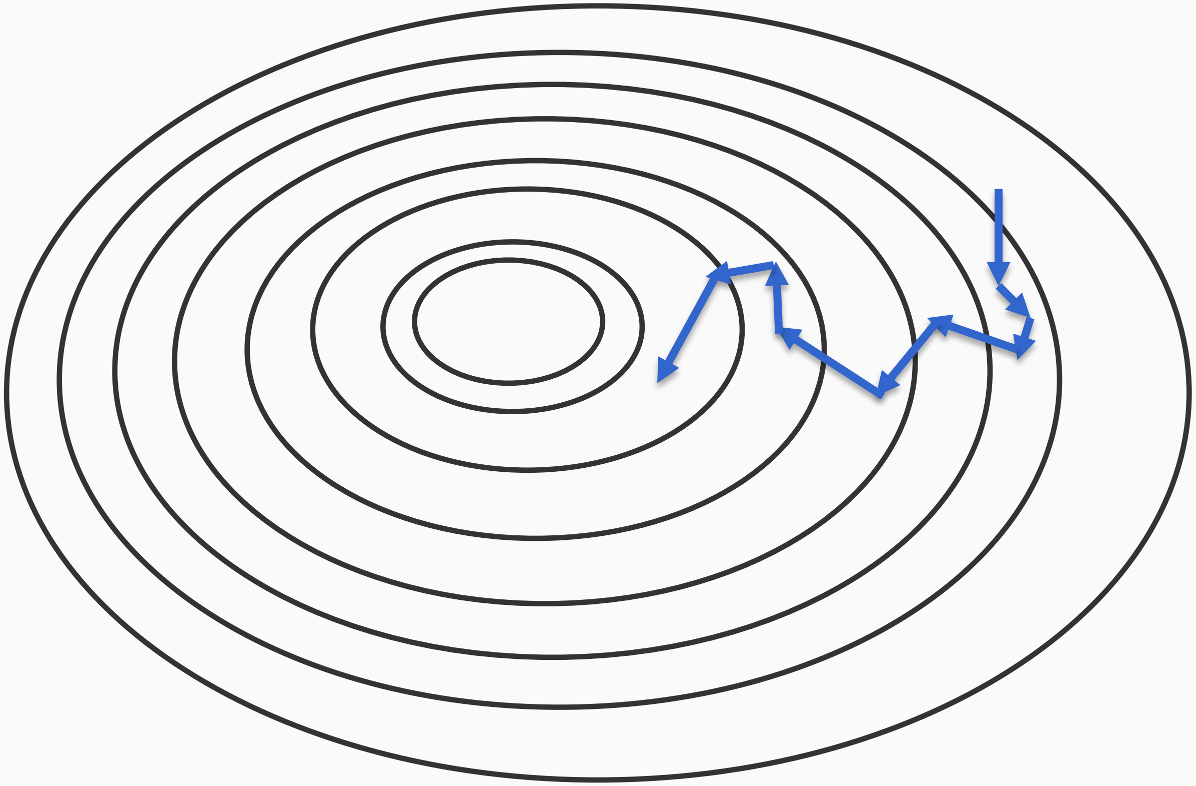

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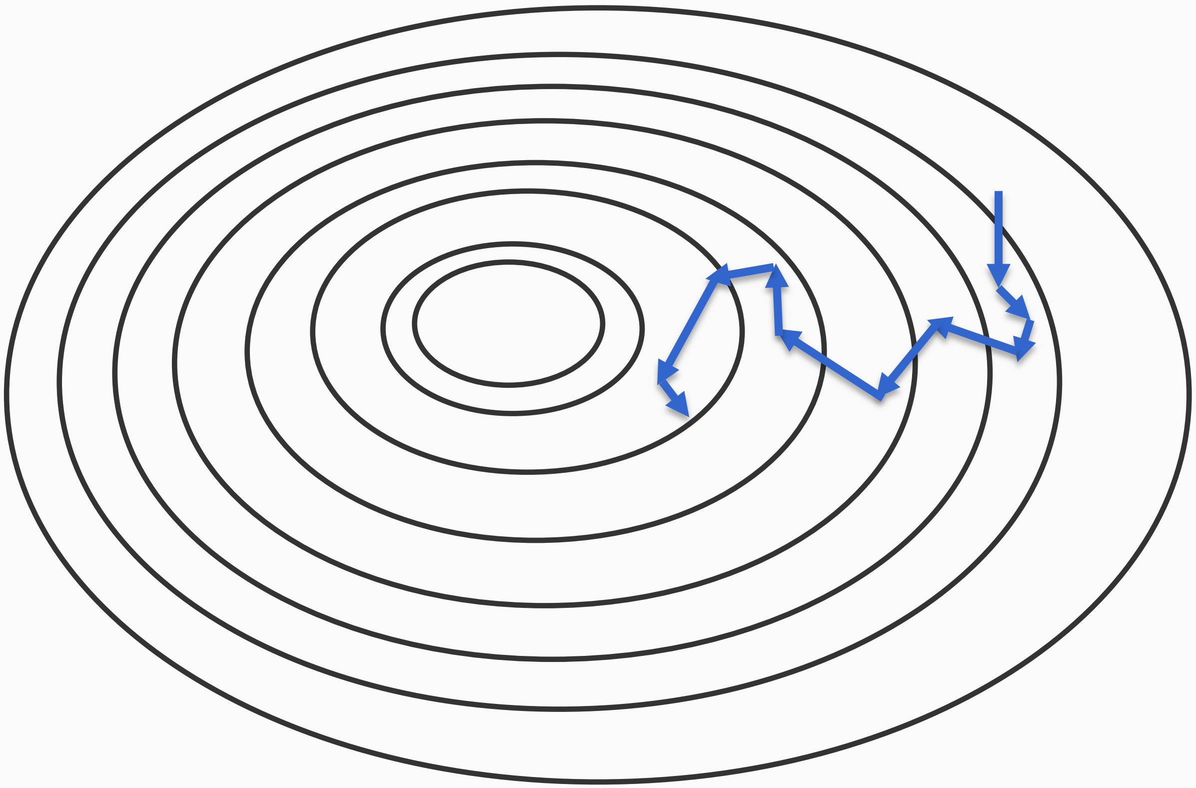

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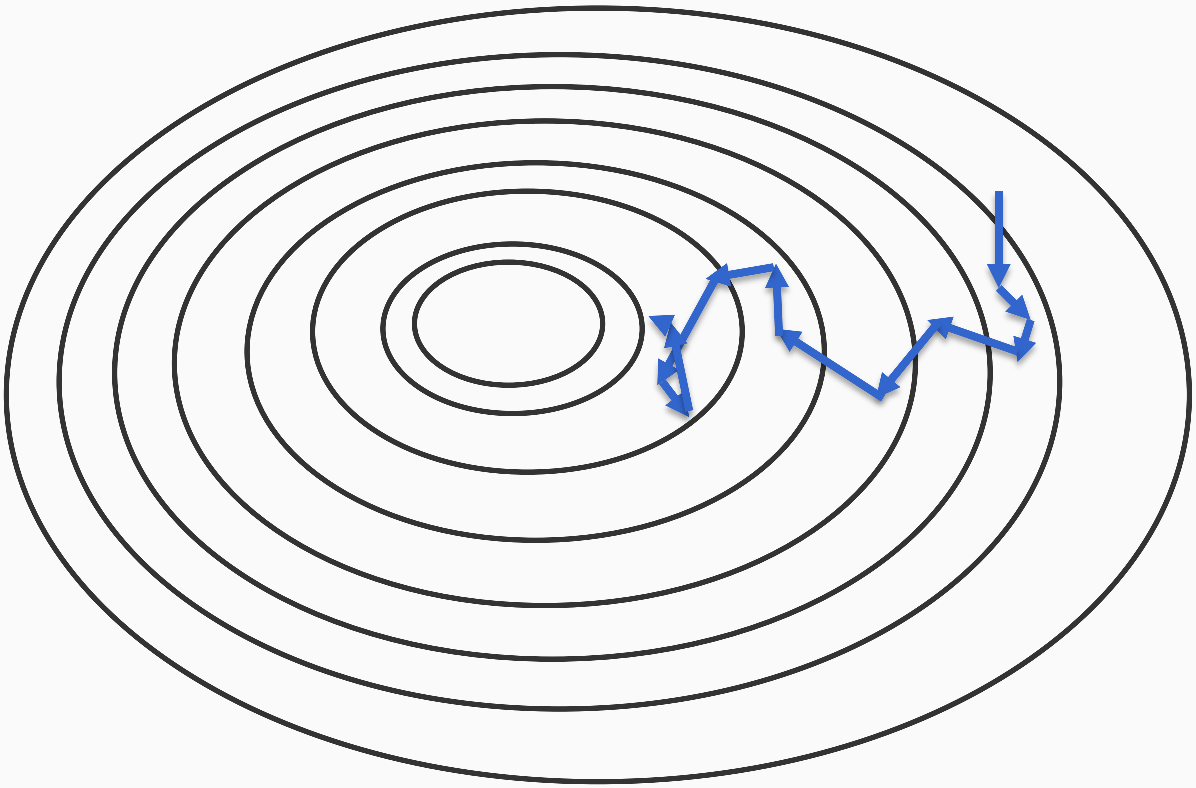

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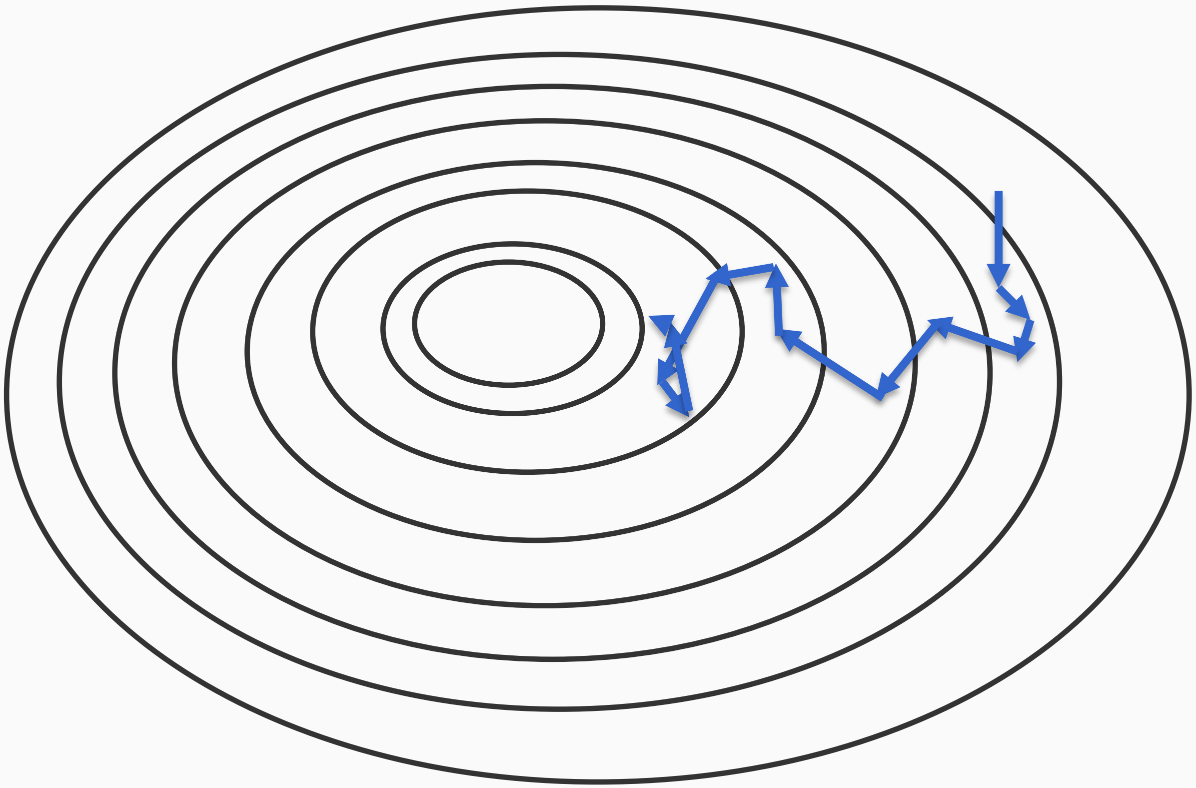

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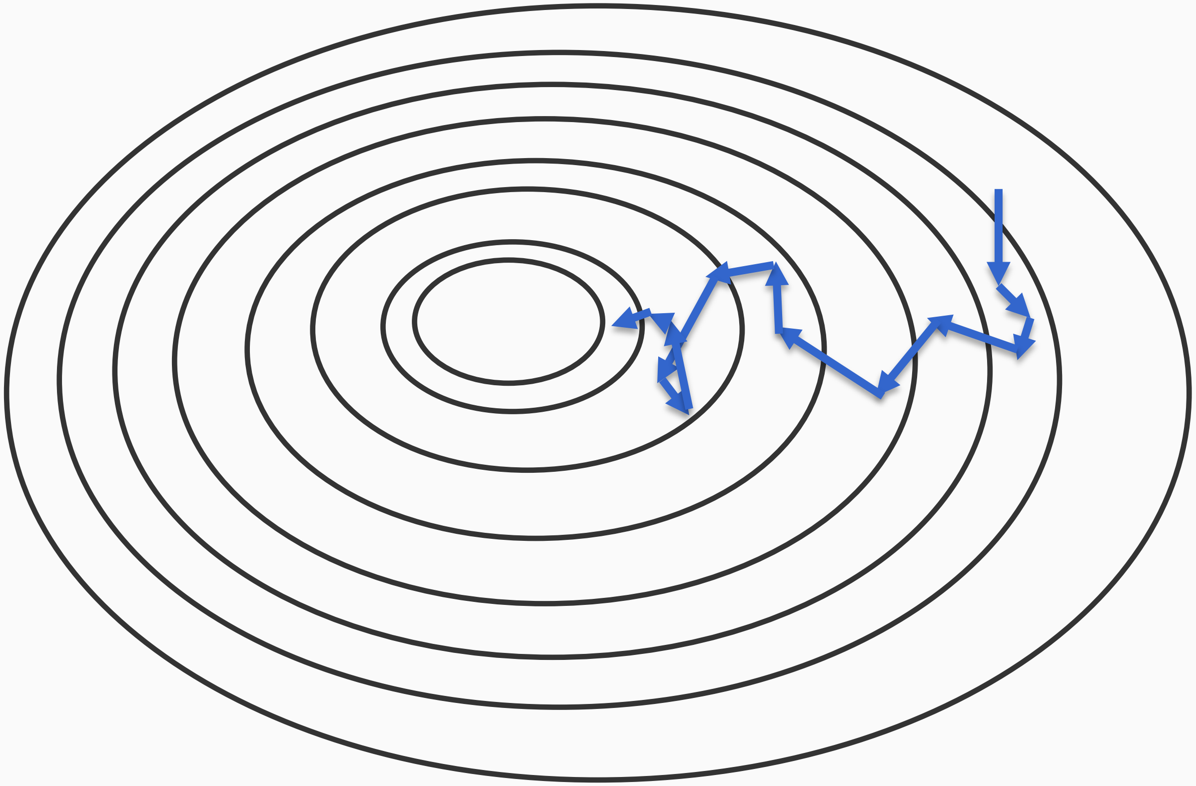

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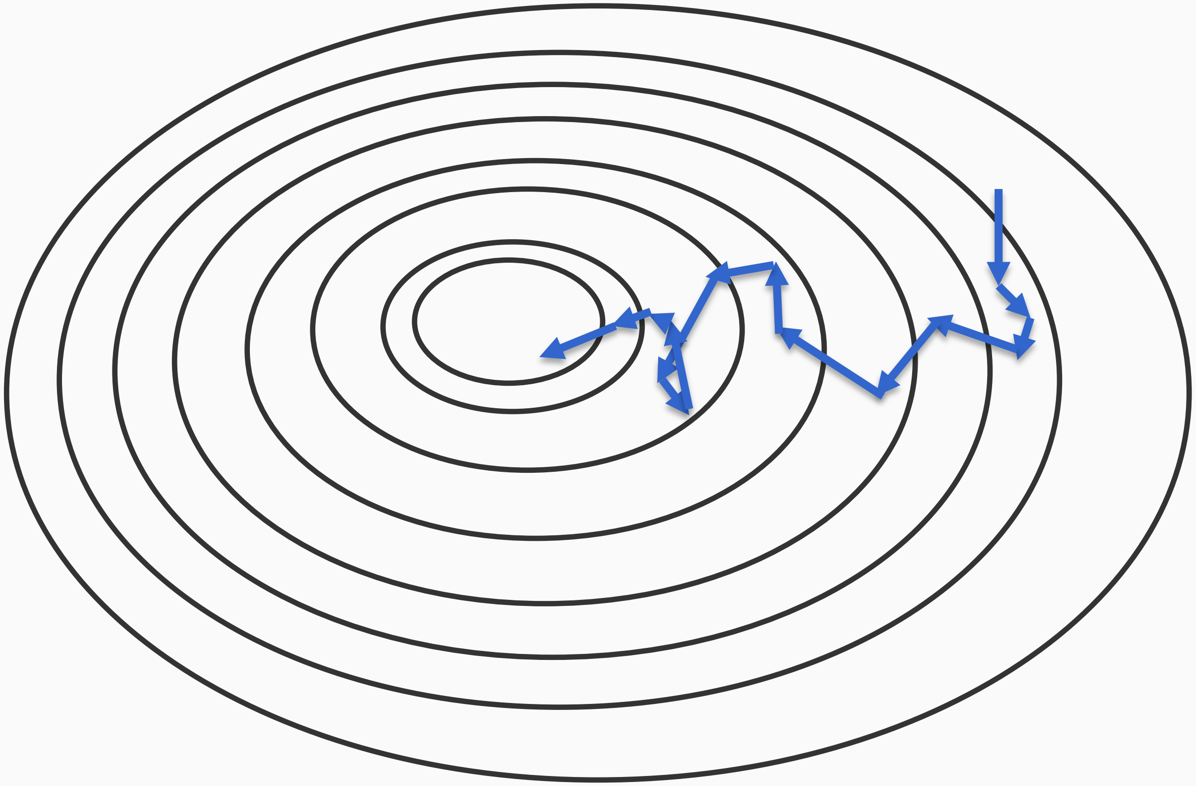

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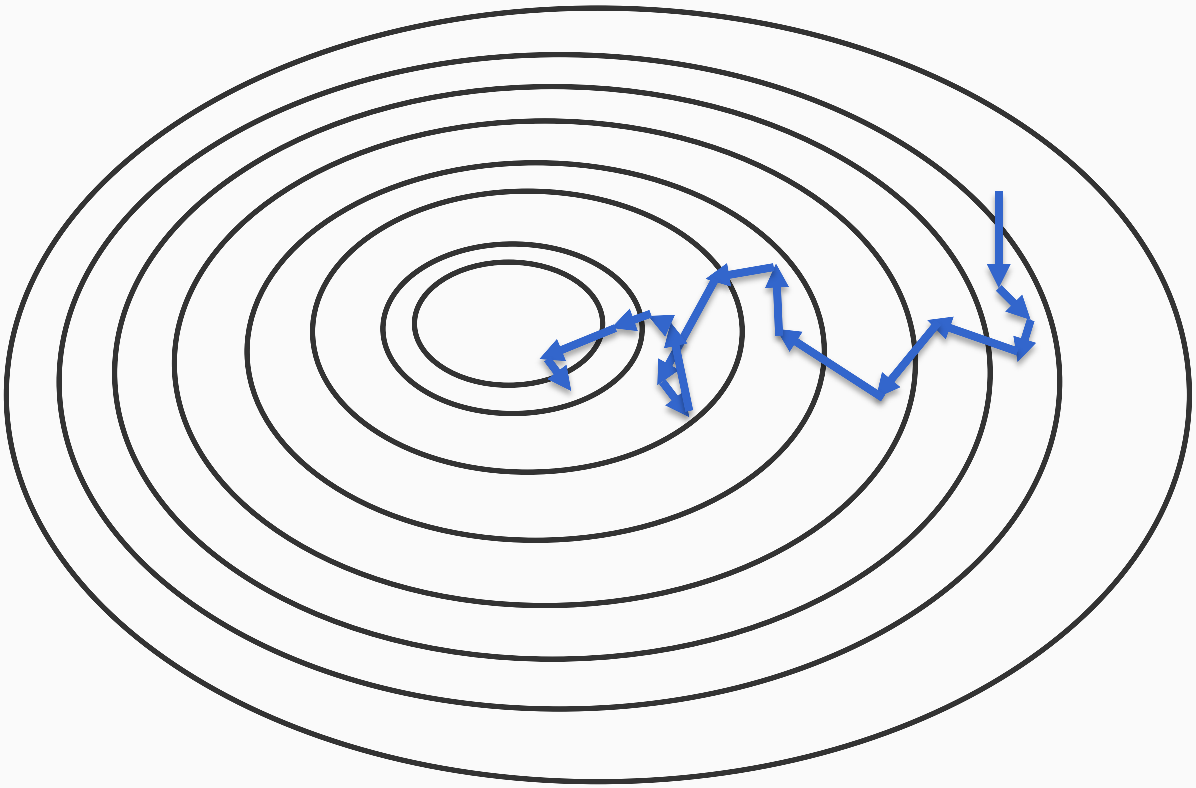

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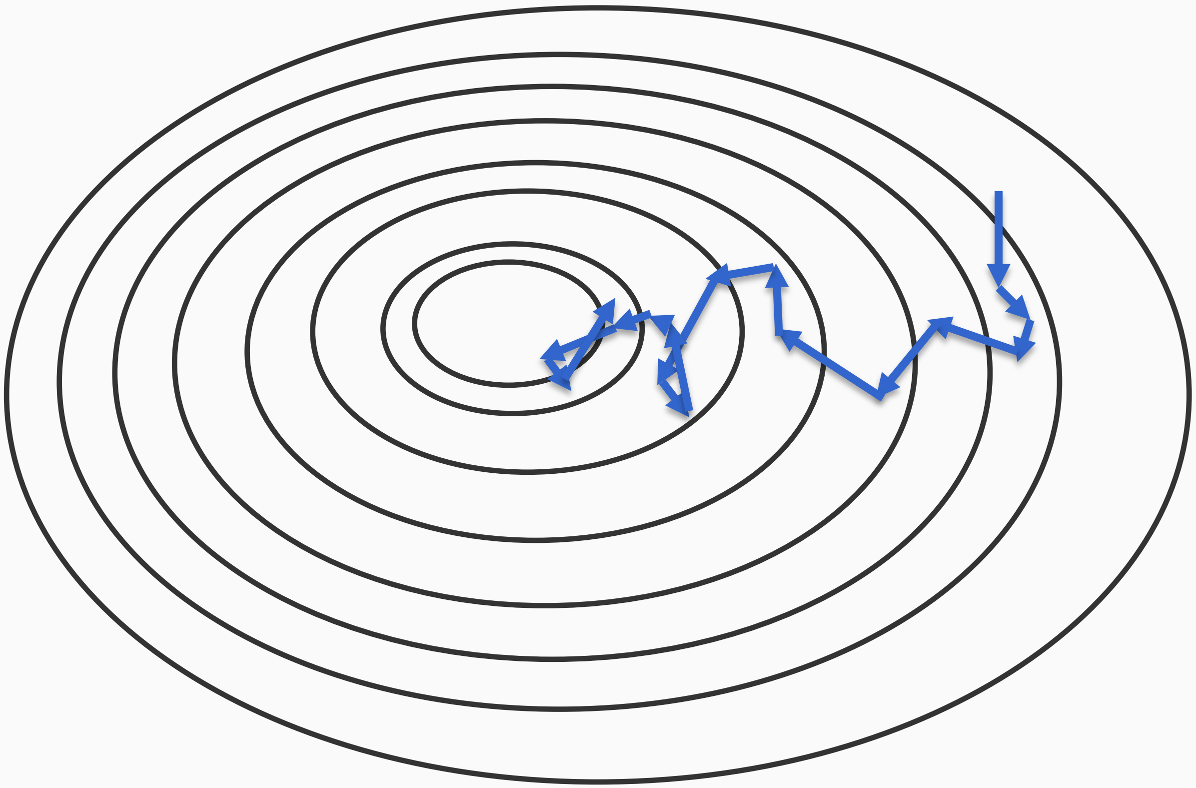

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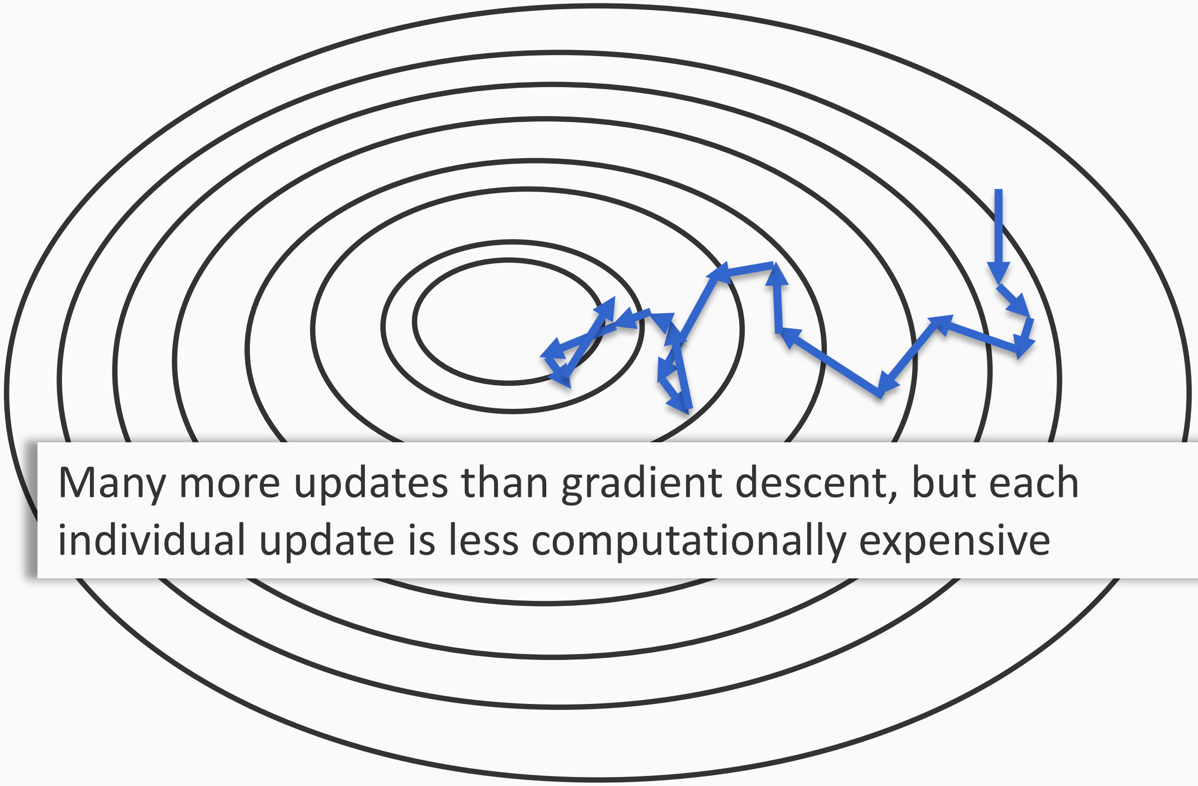

Many more updates than gradient descent, but each individual update is less computationally expensive

Stochastic Gradient descent

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Stochastic gradient descent for SVM

Given a training set S = $\{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:
    1. Pick a random example $(\mathbf{x}_i, y_i)$ from the training set S
    2. Treat $(\mathbf{x}_i, y_i)$ as a full dataset and take the *derivative of the SVM objective* at the current $\mathbf{w}^{t-1}$ to be $\nabla J^t(\mathbf{w}^{t-1})$
    3. Update: $\mathbf{w}^t \leftarrow \mathbf{w}^{t-1} - \gamma_t \nabla J^t(\mathbf{w}^{t-1})$
3. Return final **w**

What is the derivative of the hinge loss with respect to w?

(The hinge loss is not a differentiable function!)

Hinge loss is not differentiable!

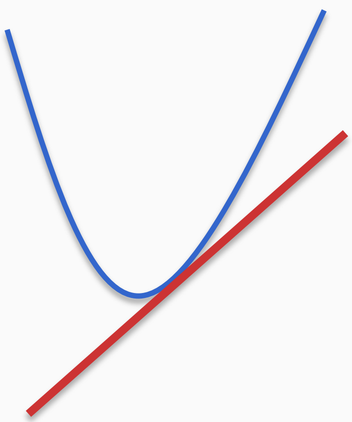
What is the derivative of the hinge loss with respect to w?

$$J^t(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C \cdot N \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

Detour: Sub-gradients

Generalization of gradients to non-differentiable functions

(Remember that every tangent lies below the function for convex functions)



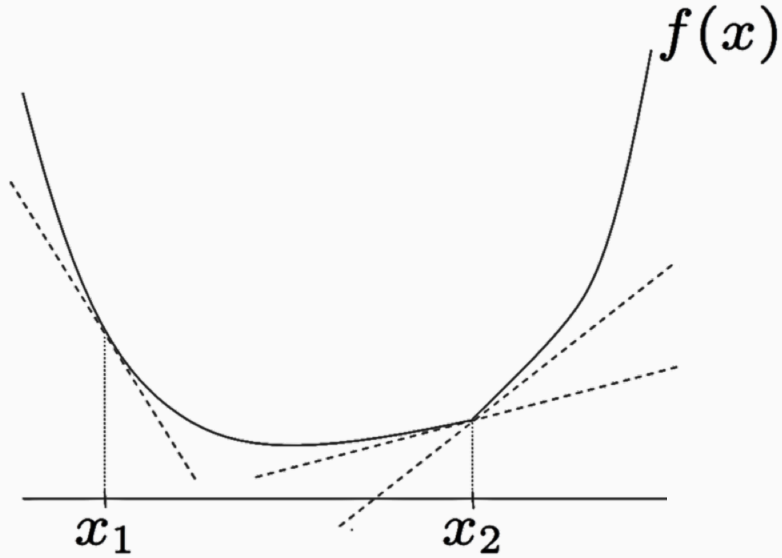
Informally, a sub-tangent at a point is any line lies below the function at the point.

A sub-gradient is the slope of that line

Sub-gradients

Formally, g is a subgradient to f at x if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$



[Example from Boyd]

Sub-gradients

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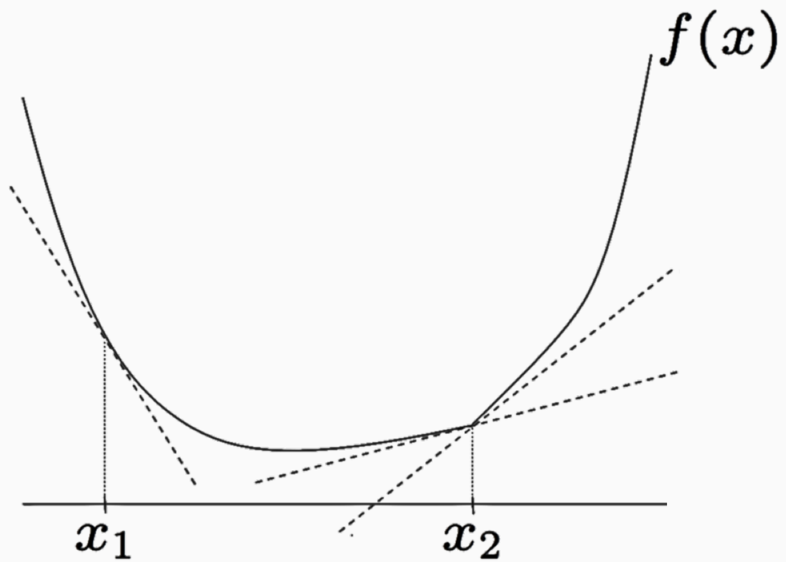
$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$

f is differentiable at x_1
Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$



g_1 is a gradient at x_1

[Example from Boyd]

Sub-gradients

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$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$

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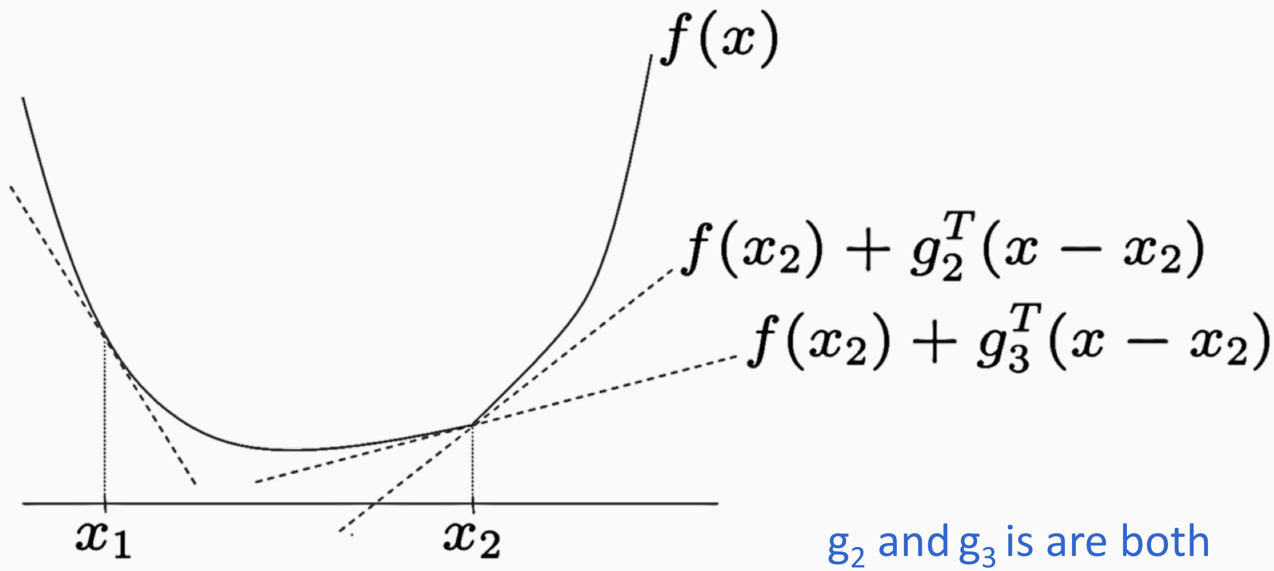

g_1 is a gradient at x_1

g_2 and g_3 is are both subgradients at x_2

[Example from Boyd]

Sub-gradient of the SVM objective

$$J^t(\mathbf{w}) = \frac{1}{2}\mathbf{w}_0^\top \mathbf{w}_0 + C \cdot N \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$$

General strategy: First solve the max and compute the gradient for each case

Sub-gradient of the SVM objective

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General strategy: First solve the max and compute the gradient for each case

$$\nabla J^t = \begin{cases} [\mathbf{w}_0; 0] & \text{if } \max(0, 1 - y_i \mathbf{w}_i^{\mathbf{x}}) = 0 \\ [\mathbf{w}_0; 0] - C \cdot N y_i \mathbf{x}_i & \text{otherwise} \end{cases}$$

Outline: Training SVM by optimization

✓ Review of convex functions and gradient descent

✓ Stochastic gradient descent

✓ Gradient descent vs stochastic gradient descent

✓ Sub-derivatives of the hinge loss

5. **Stochastic sub-gradient descent for SVM**
6. Comparison to perceptron

Stochastic sub-gradient descent for SVM

$$\nabla J^t = \begin{cases} [\mathbf{w}_0; 0] & \text{if } \max(0, 1 - y_i \mathbf{w}_i^{\mathbf{x}}) = 0 \\ [\mathbf{w}_0; 0] - C \cdot N y_i \mathbf{x}_i & \text{otherwise} \end{cases}$$

Given a training set S = $\{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$

3. Return **w**

Stochastic sub-gradient descent for SVM

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Given a training set S = $\{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:

3. Return **w**

Stochastic sub-gradient descent for SVM

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Given a training set S = {($\mathbf{x}_i$, y_i)}, $\mathbf{x} \in \Re^n$, y ∈ {-1,1}

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:
    1. For each training example ($\mathbf{x}_i$, y_i) ∈ S:

        Update $\mathbf{w} \leftarrow \mathbf{w} - \gamma_t \nabla J^t$

3. Return **w**

Stochastic sub-gradient descent for SVM

$$\nabla J^t = \begin{cases} [\mathbf{w}_0; 0] & \text{if } \max(0, 1 - y_i \mathbf{w}_i^{\mathbf{x}}) = 0 \\ [\mathbf{w}_0; 0] - C \cdot N y_i \mathbf{x}_i & \text{otherwise} \end{cases}$$

Given a training set S = $\{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:
   1. For each training example $(\mathbf{x}_i, y_i) \in S$:

      If $y_i\, \mathbf{w}^T \mathbf{x}_i \leq 1$,

      $\mathbf{w} \leftarrow (1-\gamma_t)\, [\mathbf{w}_0; \mathbf{0}] + \gamma_t\, C\, N\, y_i\, \mathbf{x}_i$

      else

      $\mathbf{w}_0 \leftarrow (1-\gamma_t)\, \mathbf{w}_0$
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Stochastic sub-gradient descent for SVM

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3. Return $\mathbf{w}$

γ_t: learning rate, many tweaks possible

Important to shuffle examples at the start of each epoch

Stochastic sub-gradient descent for SVM

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:
    1. Shuffle the training set
    2. For each training example $(\mathbf{x}_i, y_i) \in S$:

       If $y_i\, \mathbf{w}^T\mathbf{x}_i \leq 1$,

       $\mathbf{w} \leftarrow (1-\gamma_t)\, [\mathbf{w}_0; \mathbf{0}] + \gamma_t\, C\, N\, y_i\, \mathbf{x}_i$

       else

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γ_t: learning rate, many tweaks possible

Convergence and learning rates

With enough iterations, it will converge in expectation

Provided the step sizes are "*square summable, but not summable*"

- Step sizes γ_t are positive
- Sum of squares of step sizes over t = 1 to ∞ is not infinite
- Sum of step sizes over t = 1 to ∞ is infinity

- Some examples: $\gamma_t = \frac{\gamma_0}{1+\frac{\gamma_0 t}{C}}$ or $\gamma_t = \frac{\gamma_0}{1+t}$

Convergence and learning rates

- Number of iterations to get to accuracy within ϵ

- For strongly convex functions, N examples, d dimensional:
  - Gradient descent: $O(Nd \ln(1/\epsilon))$
  - Stochastic gradient descent: $O(d/\epsilon)$

- More subtleties involved, but SGD is generally preferable when the data size is huge

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6. **Comparison to perceptron**

Stochastic sub-gradient descent for SVM

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize $\mathbf{w}^0 = 0 \in \Re^n$
2. For epoch = 1 … T:
    1. Shuffle the training set
    2. For each training example $(\mathbf{x}_i, y_i) \in S$:

       If $y_i\, \mathbf{w}^T\mathbf{x}_i \leq 1$,

       $\mathbf{w} \leftarrow (1-\gamma_t)\, [\mathbf{w}_0; \mathbf{0}] + \gamma_t\, C\, N\, y_i\, \mathbf{x}_i$

       else

       $\mathbf{w}_0 \leftarrow (1-\gamma_t)\, \mathbf{w}_0$
3. Return $\mathbf{w}$

Compare with the Perceptron update:
If $y_i\mathbf{w}^T\mathbf{x}_i \leq 0$, update $\mathbf{w} \leftarrow \mathbf{w} + r\, y_i\, \mathbf{x}_i$

Perceptron vs. SVM

- Perceptron: Stochastic sub-gradient descent for a different loss
  - No regularization though

$$L_{Perceptron}(y, \mathbf{x}, \mathbf{w}) = \max(0, -y\mathbf{w}^T\mathbf{x})$$

- SVM optimizes the hinge loss
  - With regularization

$$L_{Hinge}(y, \mathbf{x}, \mathbf{w}) = \max(0, 1 - y\mathbf{w}^T\mathbf{x})$$

SVM summary from optimization perspective

- Minimize regularized hinge loss

- Solve using stochastic gradient descent
  - Very fast, run time does not depend on number of examples

  - Compare with Perceptron algorithm: similar framework with different objectives!

  - Compare with Perceptron algorithm: Perceptron does not maximize margin width
    - Perceptron variants can force a margin
- Other successful optimization algorithms exist
  - Eg: Dual coordinate descent, implemented in `liblinear`

Questions?