this determines the entire table. This is because she knows to choose 4 cups in each category, and thus each row and each column must sum to 4. The table for the general outcome looks like this:

<table>
<thead>
<tr>
<th></th>
<th>Milk First</th>
<th>Tea First</th>
</tr>
</thead>
<tbody>
<tr>
<td>Truth</td>
<td>k</td>
<td>4-k</td>
</tr>
<tr>
<td>Milk First</td>
<td>4-k</td>
<td>k</td>
</tr>
</tbody>
</table>

First, notice that correct answers are on the diagonal. So, a value of \( k \) means that the lady actually has \( 2k \) answers correct. Here the counting problem to compute the probability of a general outcome is more difficult, but it follows what is called a hypergeometric distribution. (We won’t cover this distribution in detail, but see the Wikipedia article if you want to learn more about it.) The probability becomes:

\[
p(k) = \frac{\binom{4}{k} \binom{4}{4-k}}{\binom{8}{4}}
\]

\( \checkmark \)

\( \blacksquare \)

Now, this is the probability of the lady getting exactly \( 2k \) answers correct. What we originally wanted to ask is “what is the probability of her getting this outcome or better?” To get this, we need to sum over all values \( k \) or greater (up to the max of 4). Letting \( X \) be the total number of correct answers, this is:

\[
P(\text{"2k correct or better"}) = P(X \geq 2k) = \sum_{i=k}^{4} p(i)
\]

Here are the probabilities of the 5 possible outcomes for the experiment:

- \( p(0) = \frac{1}{70} \)
- \( p(1) = \frac{16}{70} \)
- \( p(2) = \frac{36}{70} \)
- \( p(3) = \frac{16}{70} \)
- \( p(4) = \frac{1}{70} \)

Notice the symmetric in the probabilities. It is just as hard to get all wrong as it is to get all correct!

Finally, the probabilities for getting \( 2k \) correct answers or better are

\[
P(X = 0) = \frac{1}{70}, P(X = 2) = \frac{69}{70}, P(X = 4) = \frac{53}{70}, P(X = 6) = \frac{17}{70}, P(X = 8) = \frac{1}{70}.
\]

By the way, according to the legend, the lady got all 8 cups correct!

---

**Summary of General Hypothesis Test Procedure:**

1. Define the **null hypothesis**, which is the uninteresting or default explanation.

2. Assume that the null hypothesis is true, and determine the probability rules for the possible outcomes of the experiment.

3. After collecting data, compute the probability of the final outcome or even more extreme outcomes.

---

Null Hypothesis: Hypothesis that easily allows you to calculate the probability of a particular statistic.
Fair Coin Experiment

\[ \text{Exp - flips, } T = T(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \]

\( H_0 \) - coin fair, \( p = 0.5 \)

\( H_1 \) - \( p \neq 0.5 \), \( p > 0.5 \), \( p < 0.5 \)

\( n=100 \)

\[ T \sim \text{Bin}(n, p) = \text{Bin}(n, \frac{1}{2}) \]

\[ t = \sum_{i} x_i \]

\[ q \text{binom}(0.95, 100, \frac{1}{2}) = 58 \]
\[ 0.025 \quad [40, 60] \]

\[ 58 \quad 0.05 \]

\[ 0.025 \]

\[ \text{Low} \quad \{ \text{qbinom}(0.025, 100, \frac{1}{2}) \} \]

\[ \{ \text{qbinom}(0.975, 100, \frac{1}{2}) \} \]
Approx W/ Normal

\[ \Theta \sim N \left( 50, 100 \times \frac{1}{4} \right) = N(50, 25) \]

\[ z_{1-\alpha} = z_{0.95} = 1.64 \]

\[ T = \frac{\overline{x} - \mu}{\sigma / \sqrt{n}} \]

\[ (1.64 \times 5) + 50 = 58.2 \]
<table>
<thead>
<tr>
<th>Decision about null hypothesis ($H_0$)</th>
<th>Table of error types</th>
<th>Null hypothesis ($H_0$) is True</th>
<th>Null hypothesis ($H_0$) is False</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fail to reject</td>
<td></td>
<td>Correct inference (true negative) (probability = $1 - \alpha$)</td>
<td>Type II error (false negative) (probability = $\beta$)</td>
</tr>
<tr>
<td>Reject</td>
<td></td>
<td>Type I error (false positive) (probability = $\alpha$)</td>
<td>Correct inference (true positive) (probability = $1 - \beta$)</td>
</tr>
</tbody>
</table>

$P(\text{Reject} \mid H_0 \text{ true}) = \alpha$
Hypothesis Testing of Mean:

$H_0$

$\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$

If $z > z_{\alpha}$ critical, reject $H_0$.
Hyp & Mean

Reject if

\[ Z < \pm z_{\alpha/2} \]

\[ Z > z_{1 - \alpha/2} \]

\[ Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \]
2.

\[ q^* (1-\alpha) = t_{1-\alpha} \]

\[ o = ? \]

\[ t_{1-\alpha} \]

\[ T \sim S - t (dof = n-1) \]

\[ T = \frac{X - M_0}{S_n / \sqrt{n}} \]

If \( T > t_{1-\alpha} \), reject \( H_0 \).
Paired Samples – Hypothesis Testing

Same set of specimens in two conditions

"before" & "after"

\[
\begin{pmatrix}
(x_1, y_1) \\
(x_2, y_2) \\
\vdots \\
(x_n, y_n)
\end{pmatrix}
\]

\[
\begin{pmatrix}
y_1 - x_1 = h_1 \\
y_2 - x_2 = h_2 \\
\vdots \\
y_n - x_n = h_n
\end{pmatrix}
\]

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i,
\]

\[
\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i
\]
\[ h \sim N(0, \sigma^2/n) \overset{\text{IID}}{\sim} N(0, \sigma^2/n) \]

\[ T = \frac{\bar{h} - \mu}{\frac{s_n}{\sqrt{n}}} \]

\[ H_0: \text{mean of } h \text{ is zero} \]

\[ T > t_{1 - \alpha} \] for a significance level \( \alpha = 0.05 \)

If \( T > t_{1 - \alpha} \), reject \( H_0 \).
random variable about "data"

\[ T = \frac{\bar{X}_n - \mu}{S_n / \sqrt{n}} \]

\[ T \sim \text{t-dist}(\text{df} = 20) \]

\[ \text{df} = 20 = n - 1 \]

\[ t_{\alpha} = q_{t}(1-\alpha, \text{df}=20) \]

\[ P(T \leq t_{\alpha}) = 1 - \alpha \]

\[ H_0: X_i \sim N(\mu, \sigma), \sigma \text{ unknown} \]

\[ H_1: X_i \sim N(\mu', \sigma), \mu' > \mu \]

\[ p = 1 - pt(t, \text{df} = 20) \]

\[ \Pr(T \leq t) = 1 - p \]

\[ \text{critical value at } \alpha \]

"quantin"
Two sample hypothesis test

Equal variances

Scenario: Two populations, unknown means, unknown variances (equal).

\[ \sigma_x^2 = \sigma_y^2 = \sigma^2 \]

\[ x = x_1, \ldots, x_n \]

\[ y = y_1, \ldots, y_m \]
\[ S_p^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{h+m-n-2} \]

Statistic

\[ T = \frac{\bar{x}_n - \bar{y}_m}{S_p \left( \frac{1}{n} + \frac{1}{m} \right)^{\frac{1}{2}}} \]

Student with degrees of freedom.
Statistical Simulation

What is Simulation?

Why?

Complex
Predict outcomes - averaging

Monte Carlo
Random #s in Comp.

Pseudo-random #s.
Generate sequences of integers
Sequence has memory
Polynomials $\to$ chaotic
\[ \text{uniform dist} \]
Pseudo-Random.

Standard uniform distribution

\[ u \sim U(0, 1) \]

Ex: \( \text{Ber}(p) \) \( u \leftarrow \text{unif}(1) \)

\[ b = \begin{cases} 1 & \text{if } u < p \\ 0 & \text{if } u \geq p \end{cases} \]