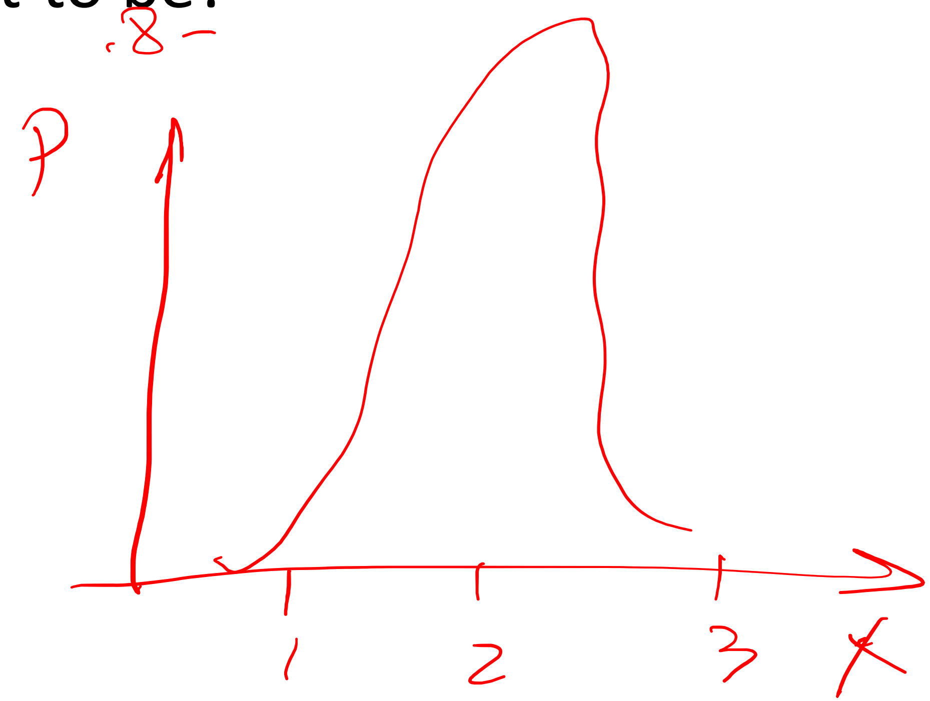
Expectation and Variance

Expectation (mean, first momentum)

If we want to summarize a random variable by a number, what value do we expect it to be?

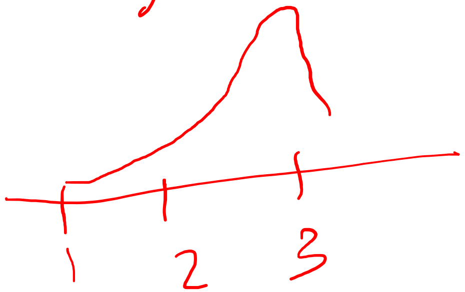


Expectation (mean)

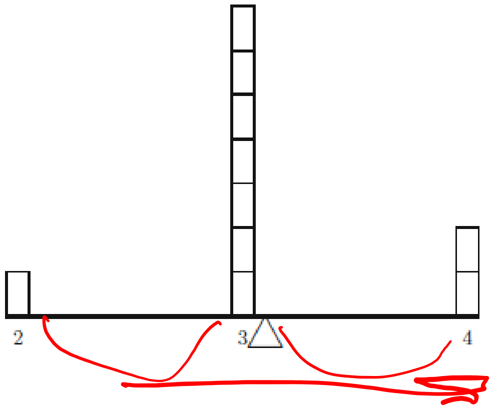
Physical interpretation:

the center of gravity of weights $p(a_i)$ placed at the points a_i

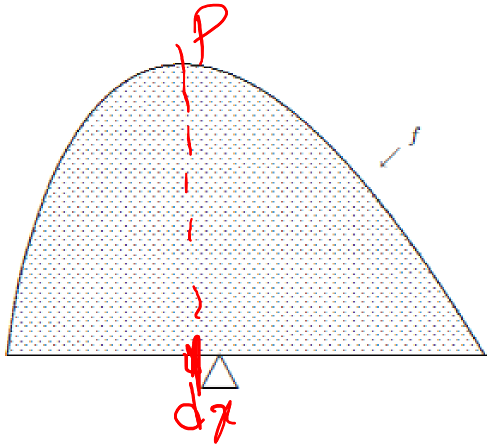
$$\frac{\sum a_i}{N}$$



discrete



continuous



$$E[X] = \sum_i a_i P(X = a_i) \qquad E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

Ex) Let X be the discrete RV that takes values 1,2,4,8,16 each with probability 1/5. what is E[X]?

1	2	$\cancel{3}$ 4	8	16
$\frac{1}{5}$	$\frac{1}{5}$			

$$E[X] = \sum_{x} x P(X = x)$$

$$= \sum_{x} x \frac{1}{5}$$

$$= \frac{1}{5}(1+2+4+8+16)$$

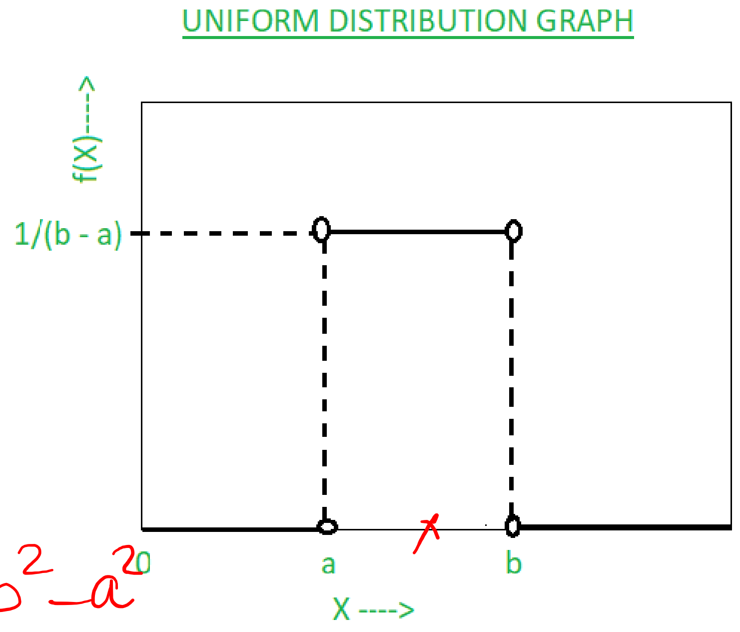
Expectation – Uniform distribution

- $P(X = x) = \frac{1}{b-a}$ $\quad\quad x \text{ in } [a, b]$
- $E[X] = \frac{a+b}{2}$

$$E[x] = \int_{-\infty}^{\infty} x \frac{1}{b-a}\, dx = \frac{1}{b-a}\int_a^b x\, dx$$

$$= \frac{x^2}{2(b-a)}\Big]_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} = a + \left(\frac{b-a}{2}\right)$$



Ex) Compute the expectation of a random variable U that is uniformly distributed over [2,5]

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad a \; b$

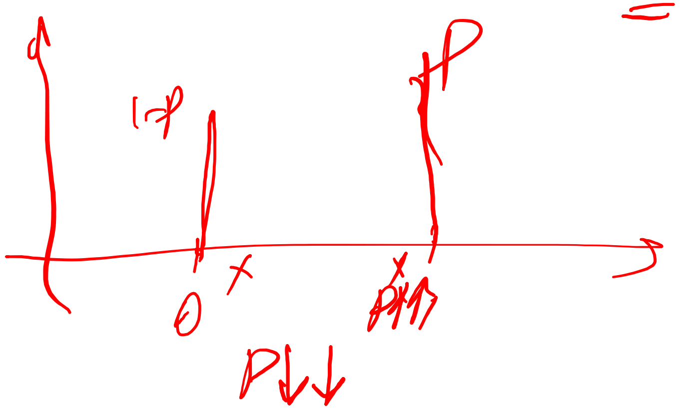
$$E[X] = \frac{2+5}{2} = 3.5$$

Expectation - Bernoulli Distribution

- $P(X = x) = p^x(1-p)^{1-x}$ $\quad$ $\underline{x \; in \; \{0,1\}}$
- $E[X] = p$

$$E[X] = \sum_x x P(X = x) = \cancel{0 \cdot P(X=0)} + 1 \cdot \overset{p}{P(X=1)}$$

$$= p$$



Expectation - Geometric Distribution

- $P(X = x) = p(1-p)^{x-1}$ 1
- $E[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{1}{p}$

$$S = \sum_{k} k\alpha^{k-1} = \frac{1}{(1-\alpha)^2} \quad 2$$

$$E[X] = \sum_{k} k\, P(X=k) \overset{1}{=} \sum_{k} k\, p\, (1-p)^{k-1} = p \sum_{k} k (1-p)^{k-1}$$

$$\overset{2}{=} p \frac{1}{(1-(1-p))^2} = p \frac{1}{p^2} = \frac{1}{p}$$

$\alpha = 1-p$

Expectation - Exponential Distribution

- $P(X = x) = \lambda e^{-\lambda x}$
- $E[X] = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}$

$$\int d(uv) = \int u\,dv + \int v\,du, \quad u = x, \quad dv = \lambda e^{-\lambda x}$$

$$\int v\,dv = \int d(uv) - \int v\,du$$

$x > 0$

$\int u\,dv$

$$E[X] = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\,dx = \int_0^\infty x\lambda e^{-\lambda x}\,dx = x(-e^{-\lambda x}) - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty$$

$$v = \int dv = \int \lambda e^{-\lambda x}\,dx = \lambda \int e^{-\lambda x}\,dx = -e^{-\lambda x}$$

$$\int v\,du = -\int e^{-\lambda x}\,dx = \frac{1}{\lambda}e^{-\lambda x} = \int v\,du$$

$$= x(-e^{-\lambda x})\Big|_0^\infty - \frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty$$

$$xe^{-\lambda x} = \frac{x}{e^{\lambda x}}$$

$$0 - 0 - \frac{1}{\lambda}(0 - e^0) = -\frac{1}{\lambda}(-1) = \frac{1}{\lambda}$$

Expectation – change of variable

$$E[g(X)] = \sum_i g(a_i) P(X = a_i)$$

$$E[g(X)] = \int g(x) f(x) dx$$

Ex) if $X \sim Ber(p)$, compute $E[\underbrace{2^X}_{g(X)}]$

$$E[X] = \sum x P(X=x)$$

$$E[g(X)] = \sum g(x) P(X=x)$$

$$E[2^X] = \sum_{x} 2^x P(X=x)$$

$$= 2^0 P(X=0) + 2^1 P(X=1)$$

$$= 1 \cdot \underbrace{P(X=0)}_{1-p} + 2\underbrace{P(X=1)}_{p} = 1-p+2p = 1+p$$

Expectation – Linearity

$$E[aX + bY] \stackrel{*}{=} aE[X] + bE[Y]$$

$$\sum_{x,y} (ax + by) P(x,y) = \sum_{x,y \in \Omega} ax P(x,y) + by P(x,y)$$

$$= \sum_{x,y} ax P(x,y) + \sum_{x,y} by P(x,y) = a \sum_x \sum_y x P(x,y) + b \sum_x \sum_y y P(x,y)$$

$$= a \sum_x x \underbrace{\boxed{\sum_y P(x,y)}}_{P(x)} + b \sum_y y \underbrace{\sum_x P(x,y)}_{P(y)} = a \underbrace{\boxed{\sum_x x P(x)}}_{E(X)} + b \underbrace{\boxed{\sum_y y P(y)}}_{E(Y)}$$

$$= aE[X] + bE[Y]$$

Ex) if we roll 10 dice and sum them up, what is the expected value of the results?

X

$10X$

$$E[10X] = \underline{10E[X]} = 10 \sum_{1}^{6} \cancel{x\,p(x)}^{1/6} = \frac{10}{6} \sum_{1}^{6} x = \frac{10}{6}(1+2+3+4+5+6)$$

$$= \frac{5}{3}(21) = 35$$

Expectation – Binomial distribution

- $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
- $E[X] = np$

$E[X] = \sum_{x} x P(X = x) =$

Variance

The spread of a random variable

$= \sum_x \bigcirc P(X=x) = \sum_x (x - E[X])^2 P(x)$

>0 >0

$g(X) = X^2$

$$Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

$$Var \geq 0$$

$$E[X^2] \geq (E[X])^2$$

$$E[X^2] \ second\ momentum$$

$$std(X) = \sqrt{Var(X)}$$