Expectation and Variance
Expectation (mean, first momentum)

If we want to summarize a random variable by a number, what value do we expect it to be?
Expectation (mean)

Physical interpretation:

the center of gravity of weights $p(a_i)$ placed at the points $a_i$

Discrete:

$$E[X] = \sum_i a_i P(X = a_i)$$

Continuous:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$
Ex) Let $X$ be the discrete RV that takes values 1, 2, 4, 8, 16 each with probability $\frac{1}{5}$. What is $E[X]$?

\[
\begin{align*}
E[X] &= \sum x P(X = x) \\
&= \sum \frac{1}{5} x \\
&= \frac{1}{5} (1 + 2 + 4 + 8 + 16) \\
&= \frac{1}{5} (31)
\end{align*}
\]
Expectation – Uniform distribution

- $P(X = x) = \frac{1}{b-a}$ for $x \in [a, b]$
- $E[X] = \frac{a+b}{2}$

$$
E[X] = \int_{a}^{b} x \cdot \frac{1}{b-a} \, dx = \frac{1}{b-a} \int_{a}^{b} x \, dx = \frac{x^2}{2} \Big|_{a}^{b} = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} = a + \frac{b-a}{2}
$$
Ex) Compute the expectation of a random variable $U$ that is uniformly distributed over $[2,5]$. 

\[
E[U] = \frac{2+5}{2} = 3.5
\]
Expectation - Bernoulli Distribution

• \( P(X = x) = p^x (1 - p)^{1-x} \) \( x in \{0,1\} \)
• \( E[X] = p \)

\[
E[X] = \sum_x x \cdot P(X = x) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = p
\]
Expectation - Geometric Distribution

\( P(X = x) = p(1 - p)^{x-1} \)  

\( E[X] = \sum_{k=1}^{\infty} kp(1 - p)^{k-1} = \frac{1}{p} \)

\[
E[X] = \sum_{k} k \cdot P(X = k) = \sum_{k} k \cdot p \cdot (1-p)^{k-1} = p \sum_{k} k \cdot (1-p)^{k-1}
\]

\[
\sum_{k} k \cdot (1-p)^{k-1} = \alpha = 1 - p
\]

\[
\frac{1}{(1 - (1-p))^2} = \frac{1}{p^2} = \frac{1}{p}
\]
Expectation - Exponential Distribution

- $P(X = x) = \lambda e^{-\lambda x}$
- $E[X] = \int_0^\infty x\lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$

\[
E[X] = \int_0^\infty x\lambda e^{-\lambda x} \, dx = \int_0^\infty \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda} 
\]
Expectation – change of variable

\[ E[g(X)] = \sum_{i} g(a_i)P(X = a_i) \]

\[ E[g(X)] = \int g(x)f(x)dx \]
Ex) if $X \sim Ber(p)$, compute $E[2^X]$

$$E[2^X] = \sum_{x} 2^x P(X = x)$$

$$E[g(x)] = \sum_{x} g(x) P(X = x)$$

$$E[2^X] = 2^0 P(X=0) + 2^1 P(X=1)$$

$$= 1 \cdot P(X=0) + 2 \cdot 2 P(X=1)$$

$$= 1 - P + 2p = 1 + p$$
Expectation – Linearity

\[ E[aX + bY] = aE[X] + bE[Y] \]

\[
\sum (\alpha x + \beta y) P(x, y) = \sum \alpha x P(x, y) + \beta y P(x, y)
\]

\[
= \sum \alpha x P(x, y) + \sum \beta y P(x, y) = a \sum x P(x, y) + b \sum y P(x, y)
\]

\[
= a \sum x \left( \frac{\sum y P(x, y)}{P(x)} \right) + b \sum y \left( \frac{\sum x P(x, y)}{P(y)} \right) = a \sum x P(x) + b \sum y P(y)
\]

\[
= a E[X] + b E[Y]
\]
Ex) if we roll 10 dice and sum them up, what is the expected value of the results?

\[
E[10X] = 10E[X] = 10 \sum_{x=1}^{x=6} \frac{1}{6} = \frac{10}{6} \sum_{x=1}^{x=6} x = \frac{10}{6} (1+2+3+4+5+6) = \frac{5}{3} (21) = 35
\]
Expectation – Binomial distribution

• $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$
• $E[X] = np$

$$E[X] = \sum_{x} x P(X = x) =$$
Variance

The spread of a random variable

\[ Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2 \]

\[ Var \geq 0 \]

\[ E[X^2] \geq (E[X])^2 \]

\[ E[X^2] \text{ second momentum} \]

\[ std(X) = \sqrt{Var(X)} \]