Introduction to Statistics

CS 3130 / ECE 3530: Probability and Statistics for Engineers

March 21, 2023

Independent, Identically Distributed RVs

Definition

The random variables $X_1, X_2, \ldots, X_n$ are said to be **independent, identically distributed (iid)** if they share the same probability distribution and are independent of each other.

Independence of n random variables means

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i).$$

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Random Samples

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A **random sample** from the distribution F of length n is a set $(X_1, \ldots, X_n)$ of iid random variables with distribution F. The length n is called the **sample size**.

- A random sample represents an experiment where n independent measurements are taken.
- A **realization** of a random sample, denoted $(x_1, \ldots, x_n)$ are the values we get when we take the measurements.

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A **statistic** on a random sample $(X_1, \ldots, X_n)$ is a function $T(X_1, \ldots, X_n)$.

Examples:

- Sample Mean

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Sample Variance

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

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Order Statistics

Given a sample $X_1, X_2, \ldots, X_n$, start by sorting the list of numbers.

- The **median** is the center element in the list if n is odd, average of two middle elements if n is even.
- The i**th order statistic** is the ith element in the list.
- The **empirical quantile** $q_n(p)$ is the first point at which p proportion of the data is below.
- **Quartiles** are $q_n(p)$ for $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The **inner-quartile range** is $IQR = q_n(0.75) - q_n(0.25)$.

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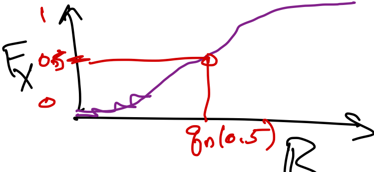

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Realizations of Statistics

Remember, a statistic is a random variable! It is not a fixed number, and it has a distribution.

If we perform an experiment, we get a realization of our sample $(x_1, x_2, \ldots, x_n)$. Plugging these numbers into the formula for our statistic gives a **realization of the statistic**, $t = T(x_1, x_2, \ldots, x_n)$.

Example: given realizations x_i of a random sample, the realization of the sample mean is $\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Upper-case = random variable, Lower-case = realization

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Statistical Plots

(See example code “StatPlots.r”)

- Histograms
- Empirical CDF
- Box plots
- Scatter plots

Sampling Distributions

Given a sample $(X_1, X_2, \ldots, X_n)$. Each X_i is a random variable, all with the same pdf.

And a statistic $T = T(X_1, X_2, \ldots, X_n)$ is also a random variable and has its own pdf (different from the X_i pdf). This distribution is the **sampling distribution** of T.

If we know the distribution of the statistic T, we can answer questions such as “What is the probability that T is in some range?” This is $P(a \leq T \leq b)$ – computed using the cdf of T.

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Sampling Distribution of the Mean

Given a sample $(X_1, X_2, \ldots, X_n)$ with $E[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$,

What do we know about the distribution of the sample mean, $\bar{X}_n$?

- It's expectation is $E[\bar{X}_n] = \mu$
- It's variance is $\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$
- As n get's large, it is approximately a Gaussian distribution with mean μ and variance σ^2/n.
- Not much else! We don't know the full pdf/cdf.

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$$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu$$

$$= \frac{1}{\cancel{n}}(n \cdot \mu)$$

$$= \mu$$

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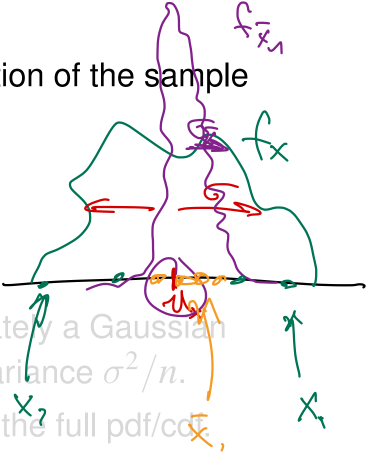

R.V. X, Y const α

$$\boxed{\mathrm{Var}[\alpha X] = \alpha^2 \mathrm{Var}[X]}$$

$$\boxed{\mathrm{Var}[X+Y]} = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X,Y)$$

X, Y independent $\mathrm{Cov}(X,Y) = 0$

$$= \mathrm{Var}[X] + \mathrm{Var}[Y]$$

$X_i \sim$ iid

$$\mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[X_i]$$

$\mathrm{Var}[X_i] = \sigma^2$

$$= \frac{1}{n^2}(n \cdot \sigma^2)$$

$$= \frac{1}{n}\sigma^2$$

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When the X_i are Gaussian

When the sample is Gaussian, i.e., $X_i \sim N(\mu, \sigma^2)$, then we know the *exact* sampling distribution of the mean $\bar{X}_n$ is Gaussian:

$$\bar{X}_n \sim N(\mu, \sigma^2/n)$$

Chi-Square Distribution

The **chi-square distribution** is the distribution of a sum of squared Gaussian random variables. So, if $X_i \sim N(0, 1)$ are iid, then

$$\boxed{Y = \sum_{i=1}^{k} X_i^2}$$

has a chi-square distribution with k **degrees of freedom**. We write $Y \sim \chi^2(k)$.

Read the Wikipedia page for this distribution!!

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Sampling Distribution of the Variance

If $X_i \sim N(\mu, \sigma)$ are iid Gaussian rv's, then the sample variance is distributed as a *scaled* chi-square random variable:

$$\frac{n-1}{\sigma^2} S_n^2 \sim \chi^2(n-1)$$

$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

Or, a slight abuse of notation, we can write:

$$S_n^2 \sim \frac{\sigma^2}{n-1} \cdot \chi^2(n-1)$$

This means that the S_n^2 is a chi-square random variable that has been scaled by the factor $\frac{\sigma^2}{n-1}$.

How to Scale a Random Variable

Let's say I have a random variable X that has pdf $f_X(x)$.

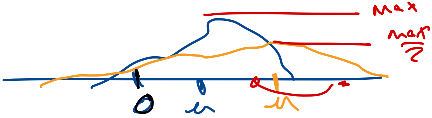
What is the pdf of kX, where k is some scaling constant?

$E[kX] = k \cdot E[X]$

The answer is that kX has pdf

$$\boxed{f_{kX}(x) = \frac{1}{k} f_X\left(\frac{x}{k}\right)}$$

See pg 106 (Ch 8) in the book for more details.



Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots$ be iid random variables from a distribution with mean μ and variance $\sigma^2 < \infty$. Then in the limit as $n \to \infty$, the statistic

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}}$$

has a standard normal distribution.

Recall $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Importance of the Central Limit Theorem

- Applies to real-world data when the measured quantity comes from the average of many small effects.
- Examples include electronic noise, interaction of molecules, exam grades, etc.
- This is why a Normal distribution model is often used for real-world data.
- Also, this “concentration of measure” effect is the basis for all of machine learning (more data, more accuracy).

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