

Notes: Expectation and Variance

CS 3130 / ECE 3530: Probability and Statistics for Engineers

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Expectation:

The **expectation of a discrete random variable** X taking values $\{a_i\}$ with probability mass function p is given by

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i).$$

The expectation is the value that you would expect on average if you repeat an experiment many times.

Example: What is the expectation of $X \sim Ber(p)$?

$$E[X] = \sum_{k=0}^1 k p(k) = 0 \cdot (1 - p) + 1 \cdot p = p$$

Example: What is the expectation of $X \sim Geo(p)$?

$$E[X] = \sum_{k=1}^{\infty} k p(k) = \sum_{k=1}^{\infty} k p(1 - p)^{k-1} = \frac{1}{p}$$

In-class Exercise: What is the expectation of a six-sided die roll?

$$E[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

The **expectation of a continuous random variable** X with probability density function f is given by

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

Example: What is the expectation of $X \sim Exp(\lambda)$?

$$E[X] = \int_0^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

Example: What is the expectation of $X \sim N(\mu, \sigma^2)$? (from book)

Linearity of expectation: If X and Y are random variables and $a, b \in \mathbb{R}$, then

$$E[aX + bY] = a E[X] + b E[Y]$$

Example: If we roll 10 dice and sum them up, what is the expected value of the result?

Answer: Let X be one die, and S the sum of 10 dice. Then, $E[S] = E[10 \cdot X] = 10 \cdot E[X] = 35$.

In-class Exercise: Remember that if $X \sim \text{Bin}(n, p)$, then X is the sum of n Bernoulli random variables, $X_i \sim \text{Ber}(p)$. Use the linearity of expectation to compute $E[X]$.

$$E[X] = E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n] = np$$

Expectation of a function $g : \mathbb{R} \rightarrow \mathbb{R}$ of a random variable:

Discrete case:

$$E[g(X)] = \sum_i g(a_i) p(a_i)$$

Continuous case:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Variance:

The **variance** of a random variable X (continuous or discrete) is given by

$$\text{Var}(X) = E[(X - E[X])^2].$$

The variance describes how “spread out” a random variable’s distribution is. The **standard deviation**, defined as the square root of the variance, is often a more useful description of the spread (it’s in the the same units as $E[X]$).

Example: The variance of Bernoulli random variable, $X \sim \text{Ber}(p)$:

$$\text{Var}(X) = \sum_{k=0}^1 p(k)(k - E[X])^2 = (1 - p)(0 - p)^2 + p(1 - p)^2 = (1 - p)p^2 + p(1 - p)^2 = p(1 - p)$$

Example: The variance of the normal distribution, $X \sim N(\mu, \sigma^2)$ (from book).

An equivalent formula for variance is

$$\text{Var}(X) = E[X^2] - E[X]^2.$$

Variance after a scaling and shift, $a, b \in \mathbb{R}$:

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

In-class exercise: What is the variance of the six-sided die roll?

$$\text{Var}(X) = \sum_{k=1}^6 \frac{1}{6} k^2 - E[X]^2 = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92$$

$$\text{Standard deviation} = \sqrt{\text{Var}(X)} \approx 1.71$$