Parameter Estimation and Error Analysis of Range Data

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Abstract

This paper discusses parameter estimation and error analysis techniques applicable to range data. Extracting surface parameters from noisy range data amounts to data reduction (parameter estimation) and error analysis, the most popular method for which is the least squares method. Most researchers usually assume that range data is Gaussian. It is argued here that linear least squares can be used profitably and with good theoretical basis regardless of the distribution of data. Using least squares, it is shown how error in range data percolates upward to manifest itself in the value of each parameter that is calculated either directly or indirectly from data. In addition to estimating surface parameters, errors in the parameters themselves are also estimated. The formalisms (with examples) for estimating error in a computed parameter, and combining two or more sets of parameter estimates into a single estimate are presented.

1 Introduction

Recently, there has been several articles dealing with parameter estimation and error analysis in range data. For example, Durrant-Whyte [4] discusses uncertain geometries in robotics. Bolle and Cooper [2] discuss ways to optimally combine estimates. All measurements contain error. Extracting surface parameters from a set of points amounts to data reduction and error analysis. The least squares analysis of data is the most popular method for these tasks. Surprisingly, however, in the computer vision literature, there is not a comprehensive description of applications of least squares methodology to range data. This paper fills that void for linear least squares.

There has been several works that use least squares analysis in range data. Faugeras and Hebert [5], for example, use least squares to extract surfaces from range data. Several aspects of least squares methodology is dealt with by Bolle and Cooper [2]. In particular, Bolle and Cooper discuss the problem of optimally combining pieces of information which is also dealt with here. They assume that range data is Gaussian. It is argued here that linear least squares can be used profitably and with good theoretical basis regardless of the distribution of data.

If a quantity is calculated using estimated parameters each of which has an associated uncertainty, then the value of the calculated quantity will also contain some degree of uncertainty. This paper shows how to calculate uncertainties in the parameters of an edge which is computed from the parameters of the two planes which intersect to form it in the first place.

2 Theory of Fitting a Plane to Noisy Range Data

The equation of a plane is given by

$$ax + by + cz = d \tag{1}$$

where

$$a^2 + b^2 + c^2 = 1 \tag{2}$$

$$d \geq 0 \tag{3}$$

and (x, y, z) is any point on the plane. The triplet (a, b, c) denotes the direction cosines of any perpendicular drawn on the plane pointing away from the origin of the coordinate system. Equivalently, it denotes the components of the unit normal vector drawn on the plane pointing away from the origin. d denotes the perpendicular distance between the plane and the origin.

2.1 Fitting a Plane to Noisy Range Data by the Eigenvector Method

The range data consist of n measurements $\{(x_i, y_i, z_i) : i = 1, 2, \ldots, n\}$. The coordinates (x_i, y_i, z_i) all contain noise. Suppose that the best fitting plane is given by $ax + by + cz = d$, where a, b, c and d satisfy conditions 2 and 3. Then the distance of the ith point to the plane without regard to sign is given by

$$d_i = (ax_i + by_i + cz_i) - (ax_p + by_p + cz_p) \tag{4}$$

where (x_p, y_p, z_p) is any point on the best fit plane. The second term of RHS of 4 is the distance from the origin to the best fitting plane, and the first term is the distance from origin to another plane parallel to the former and passing through the ith data point. According to one least squares method the best fitting plane is that plane which minimizes the non-negative quantity

$$D^2 \equiv \sum_{i=1}^{n} d_i^2 \tag{5}$$

Even before the direction cosines and d that minimize the sum are determined, one point that lies on the best fitting plane can be found. That point is the "center of mass" (x_c, y_c, z_c) of the data points:

$$x_c = \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{6}$$

$$y_c = \frac{y_1 + y_2 + \cdots + y_n}{n} \tag{7}$$

$$z_c = \frac{z_1 + z_2 + \cdots + z_n}{n} \tag{8}$$

To see this [3], suppose that the best fit plane has been found by minimizing $D^2 = \sum_{i=1}^{n} d_i^2$. Now form a coordinate system whose xy-plane is parallel to the best fit plane. Let the z_i be the z-coordinate of the ith data point, and z_0 be the z-coordinate of the best fit plane in this coordinate system. In this coordinate system, $D^2 = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n}(z_i - z_0)^2$. To minimize D^2, set

$$\frac{d\sum_{i=1}^{n}(z_i - z_0)^2}{dz_0} = 0 \qquad (9)$$

which yields

$$z_0 = \frac{\sum_{i=1}^{n} z_i}{n} \qquad (10)$$

Therefore, z_0 is the z-coordinate of the center of mass z_c of the data points. But the location of the center of mass is invariant to coordinate systems. So, the best fit plane must go through the z_c in any coordinate system, and, by similar arguments, also through x_c and y_c.

The minimization of D^2 is done with respect to a, b and c. However, a, b and c cannot be freely varied; if they could, then the result would be $a = b = c = 0$, and $D^2 = 0$. But null values for a, b and c violate the constraint of 2. So, the minimization process must be subject to constraint 2. To that end, it is necessary to introduce the Lagrange multiplier λ and minimize the quantity

$$\sum_{i=1}^{n} d_i^2 - \lambda(a^2 + b^2 + c^2) \qquad (11)$$

or, equivalently,

$$\sum_{i=1}^{n}(a\,\Delta x_{ic} + b\,\Delta y_{ic} + c\,\Delta z_{ic})^2 - \lambda(a^2 + b^2 + c^2) \qquad (12)$$

where

$$\Delta x_{ic} = x_i - x_c \qquad (13)$$

$$\Delta y_{ic} = y_i - y_c \qquad (14)$$

$$\Delta z_{ic} = z_i - z_c \qquad (15)$$

4 for d_i, and the fact that (x_c, y_c, z_c) is a point on the best fit plane are used to obtain 12. The minimization is, of course, with respect to a, b and c. Setting the derivative of 12 with respect to a equal to zero, one gets

$$\sum_{i=1}^{n}(a\Delta x_{ic} + b\Delta y_{ic} + c\Delta z_{ic})\Delta x_{ic} - \lambda a = 0 \qquad (16)$$

Taking derivatives with respect to b and c yields two more similar equations. These three equations can be combined into one matrix eigenvalue equation:

$$\begin{bmatrix} \sum\Delta x\,\Delta x & \sum\Delta y\,\Delta x & \sum\Delta z\,\Delta x \\ \sum\Delta x\,\Delta y & \sum\Delta y\,\Delta y & \sum\Delta z\,\Delta y \\ \sum\Delta x\,\Delta z & \sum\Delta y\,\Delta z & \sum\Delta z\,\Delta z \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (17)$$

where the shorthand $\sum\Delta x\,\Delta y$ stands for $\sum_{i=1}^{n}(\Delta x_{ic}\,\Delta y_{ic})$, and other elements of the 3×3 matrix, denoted by **S**, stand for similar expressions.

S is a real symmetric matrix; it must have three real eigenvalues $\lambda_k, k = 1,2,3$, and three corresponding normalized eigenvectors $\begin{bmatrix} a_k & b_k & c_k \end{bmatrix}^T, k = 1,2,3$. Identities

$$\begin{bmatrix} a_k & b_k & c_k \end{bmatrix} \begin{bmatrix} \sum\Delta x\,\Delta x & \sum\Delta y\,\Delta x & \sum\Delta z\,\Delta x \\ \sum\Delta x\,\Delta y & \sum\Delta y\,\Delta y & \sum\Delta z\,\Delta y \\ \sum\Delta x\,\Delta z & \sum\Delta y\,\Delta z & \sum\Delta z\,\Delta z \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} =$$

$$\begin{bmatrix} a_k & b_k & c_k \end{bmatrix} \lambda_k \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} \qquad (18)$$

and

$$\begin{bmatrix} a_k & b_k & c_k \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} = 1 \qquad (19)$$

yield

$$\lambda_k = \sum_{i=1}^{n}(a\,\Delta x_{ic} + b\,\Delta y_{ic} + c\,\Delta z_{ic})^2 = \sum_{i=1}^{n} d_i^2 \qquad (20)$$

20 proves that the eigenvalues of S are non-negative, and that the minimum value of $\sum_{i=1}^{n} d_i^2$ equals the lowest eigenvalue. The corresponding eigenvector determines parameters a, b and c of the best fit plane. To determine d of the plane, use the fact that (x_c, y_c, z_c) lies on that plane, and compute $d = ax_c + by_c + cz_c$. It should be noted that the other two eigenvalues correspond to stationary values of $\sum_{i=1}^{n} d_i^2$.

On some instances, one may want to find the best fit line rather than the best fit plane to a set of points [3]. There is an interesting relationship between the two problems. Both problems lead to the same eigenvalue equation! The only change is that the triplet (a, b, c) denotes the unit direction of the best fit line. The unit normal of the plane corresponds to the lowest eigenvalue. The unit direction of the line corresponds to the greatest eigenvalue. The best fit plane contains the best fit line.

For details on least squares, see [1,2,7]. Following Duda and Hart [3], the least squares method used here will be called the eigenvector method to distinguish it from other least squares methods.

2.2 Applying the Least Squares Method to Find the Best Fit Plane

The eigenvector method of finding the best fit plane is more general than the least squares method. It will be seen that the former method assumes that all measured variables contain error. In many experiments of physics and chemistry, only one variable contains error. This is the dependent variable. All other variables are independent variables which are assumed to contain no or negligible error. For example, in an experiment designed to measure the x-position of a particle as a function of time, one usually assumes that only the measurements of x are in error, not those of corresponding times. The least squares problem in these situations is relatively simple. In the present section, this method is specialized to finding the best fit plane to a set of points in 3-D space.

Assume that the data set consists of n points $\{(x_i, y_i, z_i) : i = 1, 2, \ldots, n\}$, where x_i and y_i are independent variables with no error, and z_i is the dependent variable which does contain error. Also suppose that the functional dependence of z on x and y is given by

$$z = f(x, y; p, q, r) = p + qx + ry \qquad (21)$$

Here p, q and r are parameters which will be found by the least squares procedure. In other words, from the family of infinitely many functions f each of which is characterized by a unique triplet of values of p, q and r, least squares procedure will pick out one particular f that is optimal in some sense. f describes a plane as can be seen by setting

$$d = cp \quad (22)$$

$$a = -qc \quad (23)$$

$$b = -rc \quad (24)$$

and, finally,

$$|c| = \frac{1}{\sqrt{q^2 + r^2 + 1}} \quad (25)$$

The sign of c is chosen such that d is non-negative.

With the form of f as given above, if independent variables have values x_i and y_i, then the the measured value of z should be $p + qx_i + ry_i$. However, the measured value of z is z_i. The error in the ith measurement is then

$$e_i = f(x_i, y_i; p, q) - z_i \quad (26)$$

According to the least squares method, p, q and r are to be so chosen as to minimize the following sum of the squares

$$E^2 = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (p + qx_i + ry_i - z_i)^2 \quad (27)$$

To that end, take derivatives of E^2 with respect to the parameters, and set each derivative equal to zero. This yields three simultaneous equations which are linear in the unknowns p, q and r. Combining the three equations results in the following matrix equation

$$\begin{bmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \sum z_i \\ \sum z_i x_i \\ \sum y_i z_i \end{bmatrix} \quad (28)$$

where the summation is from $i = 1$ to n. This equation can be easily solved for p, q and r. a, b, c and d can then be recovered by using 22 through 25.

2.3 Estimating Uncertainty in a Derived Quantity

Since the range data contain noise, it is reasonable to enquire about the uncertainties in the best fit parameters a, b, c and d. Unfortunately, for the eigenvector fit, this is not easy to do. However, this is easy to do for the least squares technique and modifications thereof. This is done in section 2.4. Now, how error propagates in a derived quantity is discussed [8].

Suppose that q is a quantity that is computed from m measured quantities $x_j, \ldots, x_m$. If $\overline{q}, \overline{x}_1, \ldots, \overline{x}_m$ denote the most likely values of the corresponding quantities then, in general,

$$\overline{q} = f(\overline{x}_1, \ldots, \overline{x}_m) \quad (29)$$

where $f(\cdots)$ denotes some functional dependence of $\overline{q}$ on $\overline{x}_1, \ldots, \overline{x}_m$. Now suppose that there are n measurements of the m-tuple $(x_1, , x_m)$: thus $x_{ji}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$, denotes ith measurement of the jth quantity. It follows that for each measured m-tuple, one can compute n quantities $q_i, i = 1, 2, \ldots, n$,

$$q_i = f(x_{1i}, \ldots, x_{mi}) \quad (30)$$

Then, using the the formula for the total differential,

$$\Delta q_i = \frac{\partial f}{\partial \overline{x}_1} \Delta x_{1i} + \cdots + \frac{\partial f}{\partial \overline{x}_m} \Delta x_{mi} \quad (31)$$

where $\Delta q_i = \overline{q} - q_i$ and $\Delta x_{ji} = \overline{x}_j - x_{ji}$ denote errors contained in q_i and x_{ji}, respectively. The total differential can be squared, summed over all i and the resulting equation can be manipulated to give the following approximate equation embodying standard deviations σ's

$$\sigma_q^2 = (\frac{\partial f}{\partial \overline{x}_1})^2 \sigma_{x_1}^2 + \cdots + (\frac{\partial f}{\partial \overline{x}_m})^2 \sigma_{x_m}^2 \quad (32)$$

where

$$\sigma_q^2 = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} (q_i - \overline{q})^2}{n} \quad (33)$$

$$\sigma_{x_j}^2 = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} (x_{ji} - \overline{x}_j)^2}{n} \quad (34)$$

The standard deviation σ of a quantity can be used as a measure of uncertainty in its "true" value. 32 can be used, in principle, to compute $\sigma_a^2, \sigma_b^2, \sigma_c^2$ and σ_d^2. However, applying this formula to find the standard deviations of the parameters of the best fit plane obtained using the eigenvector method is totally unmanageable. On the other hand, this formula can be applied with ease to find the standard deviations of the parameters obtained using least squares method.

2.4 Generalizing the Least Squares Method

Since, the least squares method assumes that only the dependent variable is in error, it needs to be modified to handle range data. The modification consists of weighting [7] each e_i^2 by w_i, and minimizing

$$E^2 = \sum_{i=1}^{n} w_i e_i^2 \quad (35)$$

where

$$w_i = \frac{1}{\Delta_{z_i}^2 + (\frac{\partial f}{\partial x})_{x=x_i}^2 \Delta_{x_i}^2 + (\frac{\partial f}{\partial y})_{y=y_i}^2 \Delta_{y_i}^2} \quad (36)$$

Here $\Delta_{z_i}, \Delta_{x_i}$ and Δ_{y_i} denote the errors in the corresponding data points. The formula for w_i is applicable to any f of two independent variables; for the plane, it becomes

$$w_i = \frac{1}{\Delta_{z_i}^2 + q^2 \Delta_{x_i}^2 + r^2 \Delta_{y_i}^2} \quad (37)$$

Even when the independent variables are error-free, one may still be compelled to introduce weights if each measurement of z_i has a different precision $\sigma_{z_i}^2$. Then w_i is taken to be inversely proportional to $\sigma_{z_i}^2$. This is in agreement of the appearance of the factor $\frac{1}{\sigma_{z_i}^2}$ in ??.

The matrix equation involving unknowns p, q and r becomes

$$\begin{bmatrix} \sum w_i & \sum w_i x_i & \sum w_i y_i \\ \sum w_i x_i & \sum w_i x_i^2 & \sum w_i x_i y_i \\ \sum w_i y_i & \sum w_i x_i y_i & \sum w_i y_i^2 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \sum w_i z_i \\ \sum w_i z_i x_i \\ \sum w_i y_i z_i \end{bmatrix} \quad (38)$$

or,

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{E_w} \begin{bmatrix} \sum w_i z_i \\ \sum w_i z_i x_i \\ \sum w_i y_i z_i \end{bmatrix} \quad (39)$$

where $\mathbf{E_w}$ now stands for the inverse of the modified 3×3 matrix $\mathbf{R_w}$ appearing in 38.

Using 32, the variance of p is given by

$$\sigma_p^2 = \sigma_z^2 \sum_{i=1}^{n} (\frac{\partial p}{\partial z_i})^2 \quad (40)$$

Using $\frac{\partial p}{\partial z_j} = \sum_{k=1}^{3} E_{w,1k} w_j u_{kj}$, where $u_{1j} = 1, u_{2j} = x_j$ and $u_{3j} = y_j$,

$$\sigma_p^2 = \sum_{i=1}^{n} \sigma_{z_i}^2 \sum_{l=1}^{3} E_{w,1l} w_i u_{li} \sum_{k=1}^{3} E_{w,1k} w_i u_{ki} \quad (41)$$

Assuming that $w_i \approx \frac{k}{\sigma^2_{z_i}}$, where the constant of proportionality k is often unity,

$$\sigma_p^2 = k\sum_{i=1}^{n}\frac{1}{w_i^2}\sum_{l=1}^{3}E_{w,1l}w_iu_{li}\sum_{k=1}^{3}E_{w,1k}w_iu_{ki}$$
$$= k\sum_{l=1}^{3}E_{w,1l}\sum_{k=1}^{3}E_{w,1k}\sum_{i=1}^{n}w_iu_{li}u_{ki} \quad (42)$$

Using, $\sum_{i=1}^{n} w_iu_{li}u_{ki} = R_{w,lk}$, where $R_{w,lk}$ is the lk-element of $\mathbf{R_w}$,

$$\sigma_p^2 = k\sum_{l=1}^{3}E_{w,1l}\sum_{k=1}^{3}E_{w,1k}R_{w,lk} = k\sum_{l=1}^{3}E_{w,1l}\delta_{1l} = kE_{w,11} \quad (43)$$

and, similarly,

$$\sigma_q^2 = kE_{w,22} \quad (44)$$

$$\sigma_r^2 = kE_{w,33} \quad (45)$$

If $\sigma^2_{z_i} = \sigma^2_z$, a constant, and $w_i = 1$, then $k = \sigma^2_z$.

The formulas for the standard deviations of a, b, c and d for the weighted least squares are given by

$$\frac{\sigma_d^2}{d^2} = \frac{\sigma_p^2}{p^2} + \frac{\sigma_c^2}{c^2} \quad (46)$$

$$\frac{\sigma_a^2}{a^2} = \frac{\sigma_q^2}{q^2} + \frac{\sigma_c^2}{c^2} \quad (47)$$

$$\frac{\sigma_b^2}{b^2} = \frac{\sigma_r^2}{r^2} + \frac{\sigma_c^2}{c^2} \quad (48)$$

and, finally,

$$\frac{\sigma_c^2}{c^2} = \frac{(q^2\sigma_q^2 + r^2\sigma_r^2)}{(1+q^2+r^2)^2} \quad (49)$$

These follow from 22 through 25, and 32.

There is an intuitive justification of the weight factor w_i as defined by 36. The denominator of w_i is a sum of two "variance-like" quantities. In fact, the Δ's can be replaced by the corresponding σ's in 36. The first term is the square of the error in z_i itself. The next two terms constitute the square of the total differential of f with the cross-terms neglected; thus these two terms account for the fact that z_i could be in error due to errors in x_i and y_i. The weight is chosen to be inversely proportional to the sum of the variance-like terms so that the terms with less error are weighted more heavily. This is in agreement with the appearance of the matrix $\mathbf{W}$ in the expression for E^2 in Section ??.

There is a more rigorous method for taking into account error in all variables. Let (x_i, y_i, z_i) be the measured values of variables whose true values are given by (X_i, Y_i, Z_i). Suppose that the functional dependence of Z on X, Y is known except the values of certain parameters. It is seen that the least squares is equivalent to minimizing $\sum_{i=1}^{n}(Z_i - z_i)^2 = \sum_{i=1}^{n}(f(X_i, Y_i) - z_i)^2$ with respect to parameters occurring in f. However, if there is error in all variables, the correct thing to do is to minimize

$$\sum_{i=1}^{n}(X_i-x_i)^2+\sum_{i=1}^{n}(Y_i-y_i)^2+\sum_{i=1}^{n}(Z_i-z_i)^2$$

$$= \sum_{i=1}^{n}(X_i-x_i)^2+\sum_{i=1}^{n}(Y_i-y_i)^2+\sum_{i=1}^{n}(f(X_i,Y_i)-z_i)^2 \quad (50)$$

where the RHS simply constrains x, y, z to obey the functional relationship $z = f(x, y)$. It turns out that this method of least squares for fittng a plane is equivalent to the eigenvalue method. A proof of this claim can be inferred from reference [6] where $\sum_{i=1}^{n}(X_i - x_i)^2 + \sum_{i=1}^{n}(Z_i - z_i)^2 = \sum_{i=1}^{n}(X_i - x_i)^2 + \sum_{i=1}^{n}(mX_i + b - z_i)^2$ is minimized with respect to m and b. In reference [6], the data is of the form (z_i, x_i) and $z = f(x) = mx + b$. The results can be used to show that this minimization is equivalent to minimizing $\sum_{i=1}^{n} d_i^2$ where d_i is the distance to ith data point from the straight line defined by $z = mx + b$.

2.5 Estimating Error in an Edge

Let there be two planes described by

$$a_ix + b_iy + c_iz = d_i \quad (51)$$

where $i = 1, 2$, and each of the quartets

$$(a_1, b_1, c_1, d_1) \text{ and } (a_2, b_2, c_2, d_2)$$

satisfy conditions 2 and 3. If the two planes are not parallel, that is, $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$, then they will intersect to form an edge. Let the three components of the unit direction vector of the edge be given by (l, m, n), where $l^2 + m^2 + n^2 = 1$. If $\mathbf{n}_1$ and $\mathbf{n}_2$ denote unit normals of the planes with components (a_1, b_1, c_1) and (a_2, b_2, c_2), respectively, then l, m and n are given by the unit normalized components of the cross product $\mathbf{n}_1 \times \mathbf{n}_2$. Thus,

$$l = (b_1c_2 - c_1b_2)/k \quad (52)$$

$$m = (c_1a_2 - a_1c_2)/k \quad (53)$$

$$n = (a_1b_2 - b_1a_2)/k \quad (54)$$

where $k = \sqrt{(b_1c_2 - c_1b_2)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - b_1a_2)^2}$ is the magnitude of $\mathbf{n}_1 \times \mathbf{n}_2$. If a_i, b_i and c_i are obtained from a least squares fit, then they have uncertainties associated with them. Using those uncertainties and 32, one can compute standard deviations for l, m and n:

$$\sigma_l^2 = (\frac{\partial l}{\partial a_1})^2\sigma_{a_1}^2 + (\frac{\partial l}{\partial a_2})^2\sigma_{a_2}^2 + (\frac{\partial l}{\partial b_1})^2\sigma_{b_1}^2 + (\frac{\partial l}{\partial b_2})^2\sigma_{b_2}^2$$
$$+ (\frac{\partial l}{\partial c_1})^2\sigma_{c_1}^2 + (\frac{\partial l}{\partial c_2})^2\sigma_{c_2}^2 \quad (55)$$

The formulas for σ_m^2 and σ_n^2 are obvious by symmetry.

The edge will be completely specified when one point on it is known. Assuming that the edge is not parallel to the xy-plane, that is, $n \neq 0$, one point on the edge can be taken to be on the xy-plane with coordinates, say, $(x_e, y_e, 0)$. This point must also lie on both intersecting planes, which means

$$a_1x_e + b_1y_e = d_1 \quad (56)$$

and

$$a_2x_e + b_2y_e = d_2 \quad (57)$$

Solving these two simultaneous equations for x_e and y_e, one has

$$x_e = \frac{d_1b_2 - d_2b_1}{a_1b_2 - a_2b_1} \quad (58)$$

and

$$y_e = \frac{d_1a_2 - d_2a_1}{b_1a_2 - b_2a_1} \quad (59)$$

Using the uncertainties associated with a_i, b_i, c_i and d_i, one can compute variances of x_e and z_e with the aid of 32:

$$\sigma^2_{x_e} = (\frac{\partial x_e}{\partial d_1})^2\sigma^2_{d_1} + (\frac{\partial x_e}{\partial d_2})^2\sigma^2_{d_2} + (\frac{\partial x_e}{\partial a_1})^2\sigma^2_{a_1} + (\frac{\partial x_e}{\partial a_2})^2\sigma^2_{a_2} + (\frac{\partial x_e}{\partial b_1})^2\sigma^2_{b_1} + (\frac{\partial x_e}{\partial b_2})^2\sigma^2_{b_2} \quad (60)$$

The formula for $\sigma^2_{y_e}$ is obvious by symmetry.

3 Least Squares Analysis of Synthetic Data

In this section, synthetic data that lie on a plane are generated with added Gaussian noise. Using this data, it is shown that one can *combine* the traditional least squares analysis and the eigenvector method to get error estimates of the optimal parameters.

3.1 Data

Two sets of data lying on a "roof" are obtained. The projection of the roof on any plane parallel to the zx plane is shown on Figure 1. Each data set consists of two planes each of which is a function of (x, y). The *domain* of the roof is also shown in Figure 1. z for the first plane is given by $z = 1 + x\ tan(\pi/6)$, and for the second plane, $z = (1 + 0.15\ tan(\pi/6)) - (x - 0.15)\ tan(\pi/12)$. x and y are each incremented by 0.01. Thus each plane consists of 256 3-D points. The first data set has a noise of standard deviation 0.002 added to each ordinate, and the second has a noise of standard deviation 0.01 added to each ordinate.

3.2 Estimating Parameters and Uncertai::ties

Since each variable has error, one cannot use the usual least squares method to find the parameters of two planes making up the roof. The method that treats errors in all variables in a symmetric way is the eigenvector method. This method is used to find the parameters a, b, c and d of the planes. The results are stated in Table 1.

How does one compute the uncertainties an a, b, c and d? As emphasized before, the eigenvector method is not easily amenable to finding uncertainties. So a new method is proposed here that combines the eigenvector method and the weighted least squares method of Section 2.4 to find the parameter uncertainties. In Section 2.4, each ϵ^2 is weighted by w_i. However, w_i as given by 36 is impossible to calculate for real data because the Δ's, which are exact errors of ordinates, are always unavailable. Therefore, here w_i^2 is set equal to $\frac{1}{d_i^2}$ where d_i is the distance of each point from the plane found by the eigenvector fit. Using these weights, one can proceed through the weighted least squares to obtain p, q and r and σ^2_p, σ^2_q and σ^2_r. The latter set is calculated using 43 through 45. So the question of how to estimate k arises. k is estimated this way. Note that k satisfies this equation approximately: $w_i \approx \frac{k}{\sigma_i^2} = \frac{k}{\sigma^2}$, where $\sigma = 0.002$ and 0.01 for the data sets 1 and 2, respectively. Therefore, the following approximate equation is used to calculate k

$$k \approx \frac{\sigma^2 \sum_{i=1}^{n} w_i}{n} \quad (61)$$

Once σ^2_p, σ^2_q and σ^2_r are known, $\frac{\sigma_d^2}{d^2}, \frac{\sigma_a^2}{a^2}, \frac{\sigma_b^2}{b^2}$ and $\frac{\sigma_c^2}{c^2}$ can be calculated using 46 through 49. These values, calculated using a, b, c and d of the eigenvector fit, are also listed in Table 1 below the corresponding parameter values. The values for b are not listed. They are huge because the calculated b is ≈ 0, and its exact value is 0. As expected, the data set with more noise yields larger relative uncertainties. The uncertainty entries for a appear rather large. It could be due to the *asymmetric* way x, y and z are treated in the least squares fit. This is not clear, however. Note that if a parameter is large, its uncertainty is small. In this table, values of d, a, b and c obtained using 22 through 25 are listed in parentheses for comparison against those obtained by the eigenvector method. The eigenvector method is somewhat superior.

3.3 An Edge Calculation

Using the formulas of Section 2.5, the direction and the zx-intercept of the "roof edge" can be found. The results are summarized in Table 2. The relative variances (variance/square of parameter value) are written below the corresponding parameter values. It should be noted that relative uncertainties of l and n components of this particular edge are not meaningful because calculated l and m are approximately 0, and the true values exactly equal 0. These entries tend to be quite large, and are not given in the table.

4 Concluding Remarks and Summary

Error analysis should assume a prominent role as vision systems become more sophisticated and data acquisition becomes more accurate. Already, total error of an estimated surface, which equals the quantity E^2 evaluated when $\mathbf{p}$ equals $\overline{\mathbf{p}}$, has been used for matching purposes by some workers [5].

This paper attempts to provide a methodology for performing error analysis on data obtained from a plane. It claims that least squares analysis can properly be used to analyze such data regardless of its error distribution. Using matrix algebra, it derives equations for computing the parameters that characterize the best fit plane. It also shows how the error in data percolates upward to manifest itself as uncertainties in derived parameters such as the parameters of the best fit planes, and of the edges formed by intersection of these planes. It also shows how one can optimally combine two or more estimates of a set of parameters of a surface into a single set. It presents all the equations necessary to compute the above quantities, and illustrates their use by examples.

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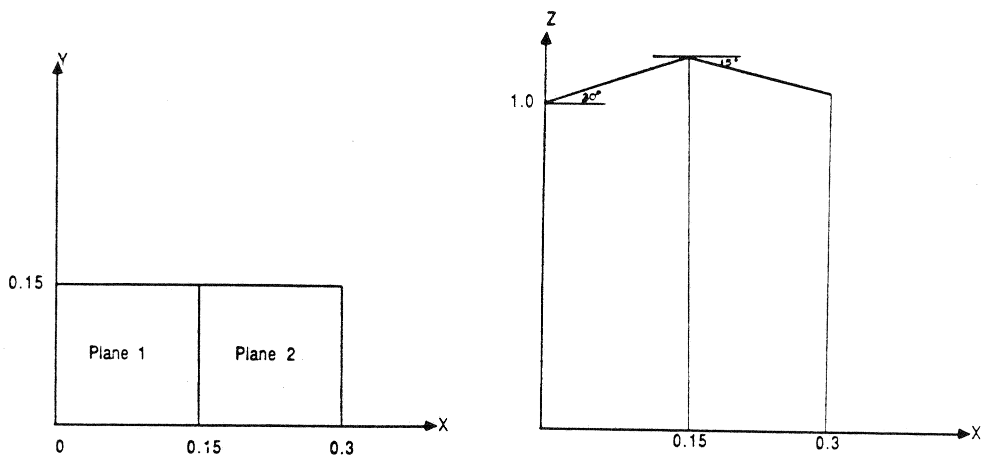

Figure 1: The domain, and the projection of the roof on the xz-plane.

Table 1: Parameters of the "roof": true and estimated values from the eigenvector and least squares fits. E = exact, D1 = data set 1 and D2 = data set 2.

	Std. dev.	Plane 1 *a*	*b*	*c*	*d*	Plane 2 *a*	*b*	*c*	*d*
E	0.000	-0.5000	0.0000	0.8660	0.8660	0.2588	0.0000	0.9659	1.0884
D1	0.002	-0.4998	0.0017	0.8662	0.8667	0.2597	0.0013	0.9657	1.0890
		0.210		0.004	0.004	0.002		0.000	0.000
		(-0.5021)	(-0.0042)	(0.8649)	(0.8643)	(0.2599)	(0.0026)	(0.9656)	(1.0892)
D2	0.01	-0.4996	0.0103	0.8662	0.8688	0.2588	0.0095	0.9659	1.0921
		0.481		0.006	0.006	0.044		0.000	0.000
		(-0.4389)	(0.0802)	(0.8949)	(0.9047)	(0.2458)	(0.0175)	(0.9692)	(1.0944)

Table 2: Parameters of the "roof edge": true values and estimated values along with relative uncertainties.

	Std. dev. of error	l	m	n	x_e	z_e
E	0.000	0.0000	1.0000	0.0000	0.15	1.0866
D1	0.002	0.0007	0.9999	-0.0011	0.1503	1.0873
			0.000		0.177	0.000
D2	0.01	0.0024	0.9999	-0.0075	0.1511	1.0902
			0.000		0.219	0.000