doing its work), ... $\}$). Solving for $\Pr\{m_i > 0\}$ requires finding the constant A' in (7.80). In fact, the major difference between open and closed networks is that the relevant constants for closed networks are tedious to calculate (even by computer) for large networks and large M.

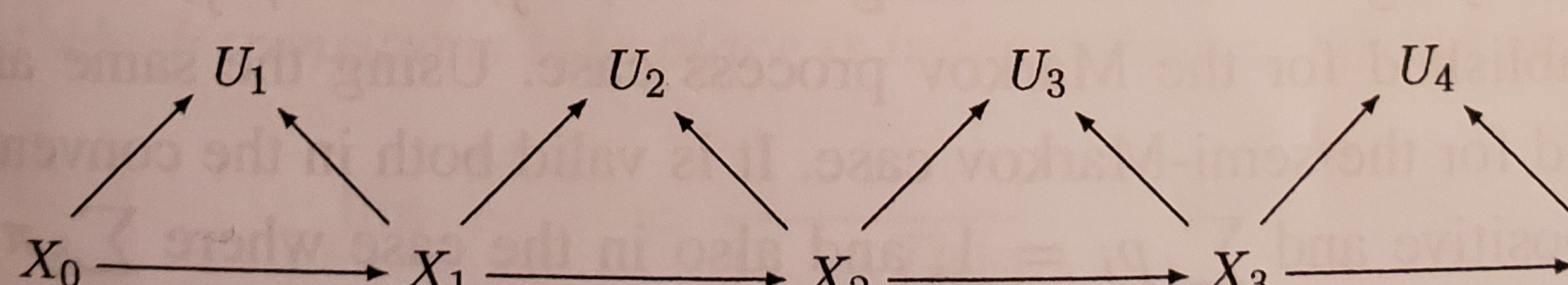
Semi-Markov processes

Semi-Markov processes are generalizations of Markov processes in which the time intervals between transitions have arbitrary distributions rather than exponential distributions. To be specific, there is an embedded Markov chain, $\{X_n;\ n \geq 0\}$ with a finite or countably infinite state space, and a sequence $\{U_n;\ n \geq 1\}$ of holding intervals between state transitions. The epochs at which state transitions occur are then given, for $n \geq 1$, as $S_n = \sum_{m=1}^{n} U_n$. The process starts at time 0 with S_0 defined to be 0. The semi-Markov process is then the continuous-time process $\{X(t);\ t \geq 0\}$ where, for each $n \geq 0$, $X(t) = X_n$ for t in the interval $S_n \leq X_n < S_{n+1}$. Initially, $X_0 = i$ where i is any given element of the state space.

The holding intervals $\{U_n;\ n \geq 1\}$ are non-negative rv s that depend only on the current state X_{n-1} and the next state X_n. More precisely, given $X_{n-1} = j$ and $X_n = k$, say, the interval U_n is independent of $\{U_m;\ m < n\}$ and independent of $\{X_m;\ m < n-1\}$ The conditional CDF for such an interval U_n is denoted by $G_{jk}(u)$, i.e.,

$$\Pr\{U_n \leq u \mid X_{n-1} = j, X_n = k\} = G_{jk}(u). \tag{7.87}$$

The dependencies between the rv s $\{X_n;\ n \geq 0\}$ and $\{U_n;\ n \geq 1\}$ are illustrated in Figure 7.17.



The statistical dependencies between the rv s of a semi-Markov process. Each holding interval U_n, conditional on the current state X_{n-1} and next state X_n, is independent of all other states and holding intervals. Note that, conditional on X_n, the holding intervals $U_n, U_{n-1}, \ldots$ are statistically independent of $U_{n+1}, X_{n+2}, \ldots$.

The conditional mean of U_n, conditional on $X_{n-1} = j, X_n = k$, is denoted $\overline{U}(j,k)$, i.e.,

$$\overline{U}(j,k) = \mathsf{E}\left[U_n \mid X_{n-1} = j, X_n = k\right] = \int_{u \geq 0} [1 - G_{jk}(u)]du. \tag{7.88}$$

A semi-Markov process evolves in essentially the same way as a Markov process. Given an initial state, $X_0 = i$ at time 0, a new state $X_1 = j$ is selected according to the embedded chain with probability P_{ij}. Then $U_1 = S_1$ is selected using the distribution $G_{ij}(u)$. Next a new state $X_2 = k$ is chosen according to the probability P_{jk}; then, given $X_1 = j$ and $X_2 = k$, the interval U_2 is selected with CDF $G_{jk}(u)$. Successive state transitions and transition times are chosen in the same way.

The steady-state behavior of semi-Markov processes can be analyzed in virtually the same way as that of Markov processes. We outline this in what follows, and often omit proofs where they are the same as the corresponding proof for Markov processes. First, since the holding intervals, U_n, are rv s, the transition epochs, $S_n = \sum_{m=1}^{n} U_n$, are also rv s. The following lemma then follows in the same way as Lemma 7.2.3 for Markov processes.

Lemma 7.8.1 *Let $M_i(t)$ be the number of transitions in a semi-Markov process in the interval $(0,t]$ for some given initial state $X_0 = i$. Then $\lim_{t\to\infty} M_i(t) = \infty$ WP1.*

In what follows, we assume that the embedded Markov chain is irreducible and positive recurrent. We want to find the limiting fraction of time that the process spends in any given state, say j. We will find that this limit exists WP1, and will find that it depends only on the steady-state probabilities of the embedded Markov chain and on the expected holding interval in each state. This is given by

$$\overline{U}(j) = \mathsf{E}\left[U_n \mid X_{n-1} = j\right] = \sum_k P_{jk}\mathsf{E}\left[U_n \mid X_{n-1} = j, X_n = k\right] = \sum_k P_{jk}\overline{U}(j,k), \tag{7.89}$$

where $\overline{U}(j,k)$ is given in (7.88). The steady-state probabilities $\{\pi_i;\ i \geq 0\}$ for the embedded chain tell us the fraction of transitions that enter any given state i. Since $\overline{U}(i)$ is the expected holding interval in i per transition into i, we would guess that the fraction of time spent in state i should be proportional to $\pi_i\overline{U}(i)$. Normalizing, we would guess that