The Constraint Face Maximum Volume Ellipsoid Method: A Geometric Approach to Satisfiability

Thomas C. Henderson
University of Utah

UUCS-24-XXX

Kahlert School of Computing
University of Utah
Salt Lake City, UT 84112 USA

10 October 2023

Abstract

SAT can be posed as a geometric problem[8, 9, 7]. The 2^n SAT models are represented as the vertexes of the $n-D$ hypercube, H_n. Each disjunction in a Conjunctive Normal Form sentence defines a set of models which falsifies the sentence; the set of corresponding excluded vertexes comprises a sub-hypercube of H_n. A hyperplane can be defined which separates this sub-hypercube from the other vertexes. The intersection of the non-negative half-spaces of these hyperplanes produces a convex polytope called the feasible region, F, which if satisfiable contains vertexes of H_n

The hyperplanes generated from the disjunctions are $(n-1)$-dimensional constraint faces for F, and each solution vertex is the intersection of some subset of these. A high-level description of the Constraint Face MVE Method is as follows:

> For each constraint face, create a parallel constraint face very close to it and with opposite normal; find the new feasible region, F', with this constraint face. Now find the MVE for F' – the ellipsoid axes can be used pairwise to trace a set of boundary points on F. If there is a solution vertex, some point in this trace will be close enough to the vertex to determine that there is a SAT solution. If no such point is found from the traces of any constraint face, then the CNF sentence is not satisfiable.

1 Introduction

SAT can be defined as follows: Given a logical sentence over n variables, determine if there is an assignment of truth values to the variables which makes the sentence true. Note that for an n-variable sentence, there are 2^n possible complete truth assignments (also called models). The complete set of models (or complete conjunction set) can be mapped onto the vertexes of a unit n-dimensional hypercube, called H_n in a straightforward way: the truth assignment values serve as the coordinates in n-dimensional space. Assume the logical sentences are represented as CNF sentences, i.e., a conjunction of disjunctions of literals. Then any assignment of truth values which makes a disjunction false renders the CNF sentence false. If the disjunction has k literals, then there is one truth assignment to the atoms of these literals which makes the disjunction false; however, the variables not in the disjunction can take on either truth value, and so there are 2^{n-k} complete conjunctions which make the sentence false. This set is in fact a sub-hypercube of H_n.

The method presented here shows how each disjunction in the CNF sentence gives rise to a hyperplane which separates the non-solution vertexes (on the negative side of the hyperplane) of H_n from the solution vertexes (on the non-negative side of the hyperplane); i.e., the intersection of the non-negative half-spaces of these hyperplanes results in a convex feasible region which must contain any solution which exists.

2 Background

For a detailed discussion of the SAT problem and its complexity, see [10]. Related work on a geometric approach started with Gomory [6] who sought integer solutions for linear programs. Given the semantics of the literals in a disjunction, a linear inequality can be formed summing x_i for atoms in the clause and $(1 - x_i)$ for negated atoms in the clause and setting this to be greater than or equal to 1. Next, a $\{0, 1\}$ solution is sought resulting in an integer linear programming problem. If a non-$\{0, 1\}$ solution is found, Gomory proposed a way to separate (via a *cutting plane*) that solution from all integer solutions. This method has been used in finding lower complexity ways to provide theorems for proving the boundedness of polytopes, cutting plane proofs for unsatisfiable sentences, pseudo-Boolean optimization, etc. (see [1, 2, 3, 4, 5]). The *Chop-SAT* method has been proposed as a way to solve SAT and PSAT [8, 9, 7]. The *Chop-SAT* method was developed independently of Gomory and Chvatal's work, and is based on fundamentally different geometric insights.

3 The Geometric Approach

In the earlier work cited above, SAT is cast as a linear programming problem:

Minimize $\mathbf{f}^T x$

Subject to: $Ax \leq c$

where each constraint is given by:

$-\alpha_i \cdot x \leq c_i$

A solution for the SAT sentence exists iff a solution exists for the LP problem such that every component of x has a value equal to 0 or 1.

Given a set of m conjuncts, $C_i, i = 1 : m$, each conjunct is used to produce a hyperplane of dimension $n - 1$ which separates the solutions (i.e., some subset of vertexes of H_n) from non-solutions. The hyperplane for the i^{th} conjunct is:

$\alpha(i) \cdot x + c = 0$

Each of these hyperplanes produces an inequality:

$-\alpha(i) \cdot x \leq c_i$

for which the signed distance of a point is used to separate solution from non-solution vertexes. A matrix, A, is produced where each row is the $1 \times n$-tuple $\alpha(i)$, *the unit normal to the hyperplane*. An $n \times 1$ vector, c, is constructed where the i^{th} element of c is c_i. The original $(n-1) - D$ faces of H_n are added to the set of constraints to ensure boundedness.

The way these hyperplanes are constructed, it is now possible to run the interior-point method for linear programming to find feasible points which minimize $f^T x$ for $x \in \mathcal{F}$, where $\mathcal{F}$ is the feasible region and f is a unit vector in the desired projection direction. Note that if neither of the projection onto the positive and negative directions of some basis vector results in a 0 or 1 value, respectively, then the CNF has no solution. However, there are feasible regions for unsatisfiable sentences which do have such 0,1 projections, so this is a sufficient but not necessary condition.

4 Chop-SAT

Given m conjuncts, $C_i, i = 1 \ldots m$, then let:

$C_i = L_1 \vee L_2 \vee \ldots \vee L_k$

Note that any complete truth assignment with $\neg L_1 \wedge \neg L_2 \wedge \ldots \wedge \neg L_k$ makes C_i false.

Observe that:

- If $k = n$, then this eliminates 1 solution ($H_0 \equiv$ 0-D vertex).
- If $k = n - 1$, then this eliminates 2 solutions ($H_1 \equiv$ 1-D segment).
- If $k = n - 2$, then this eliminates 4 solutions ($H_2 \equiv$ 2-D square).
- ...
- If $k = 1$, then this eliminates half the solutions in the hypercube (H_{n-1}).

The individual hyperplane is determined as follows. Let $A = \{1, 2, \ldots, n\}$ indicate the atoms, and $\mathcal{I} \subseteq A$. Given $C_i = L_1 \vee L_2 \vee \ldots \vee L_k$, then define α_i, the hyperplane normal vector, as follows.

$$\forall i_j \in \mathcal{I}, \alpha_i(i_j) = 1 \textit{ if } L_j \textit{ is an atom } a_{i_j}, \textit{ else } -1$$

$$\forall m \notin \mathcal{I}, \alpha_i(m) = 0$$

$$\alpha_i = \frac{\alpha_i}{\| \alpha_i \|}$$

In order to get the constant for the hyperplane equation, a point must be found on the hyperplane. This is selected so that the hyperplane cuts the edges of the hypercube at a distance ξ from the non-solution vertex. This distance depends on the number k of literals:

$$d = \| \xi \frac{\overline{\mathbf{b_k}}}{k} \|$$

where $\overline{b_k}$ is a k-tuple of 1's. Next:

$$\forall i_j \in \mathcal{I}, p(i_j) = 0 \textit{ if } L_i \textit{ is an atom, else } 1$$

$$\forall m \notin \mathcal{I}, p(m) = 0$$

Then p is a non-solution vertex. To find a point, q, on the hyperplane:

$$q = p + d\alpha_i$$

This allows a solution for the constant, c, in the hyperplane:

$$c_i = -(\alpha_i \cdot q)$$

This yields the hyperplane equation:

$$\alpha_i \cdot x + c = 0$$

and the resulting inequality:

$$-\alpha_i \cdot x \leq c$$

5 Conclusions and Future Work

The Geo-SAT approach has been extended to non-Euclidean geometry, and first results indicate that solutions can be found. However, a number of things remain to be done:

1. Determine the convergence properties of the Barrier method in non-Euclidean space.
2. Perform a systematic study of solutions finding behavior in dimensions higher than 3.
3. Explore methods to speed up the discovery of a solution; e.g., as a force path is followed, project the path points onto the appropriate constraint surfaces and determine whether the direction of motion of the projected points can be used to move to a solution vertex more rapidly.

References

[1] S.R. Buss and P. Clote. Cutting Planes, Connectivity, and Threshold Logic. *Archive for Mathematic Logic*, 35:33–62, 1996.

[2] V. Chvatal. Edmonds Polytopes and a Hierarchy of Combinatorial Problems. *Discrete Mathematics*, 4:305–337, 1973.

[3] V. Chvatal. Cutting Planes in Combinatorics. *European Journal of Combinatorics*, 6:217–226, 1985.

[4] W. Cook, C.R. Coullard, and G. Turan. On the Complexity of Cutting-Plane Proofs. *Discrete Applied Mathematics*, 18:25–38, 1987.

[5] J. Devriendt, S. Gocht, E. Demirovic, J. Nordstrom, and P.J. Stuckey. Cutting to the Core of Psuedo-Boolean Optimization: Combining Core-Guided Search with Cutting Planes Reasoning. In *Thirty-Fifth AAAI Conference on Artificial Intelligence*. Elsevier, 2021.

[6] R.E. Gomory. Outline of an Algorithm for Integer Solution to Linear Programs. *Bulletin of the American Mathematical Society*, 64(5):275–278, 1958.

[7] T.C. Henderson, D. Sacharny, X. Fan, A. Mitiche, and T. Geary. Geo-SAT: A Geometric Approach to Satisfiability. Technical Report UUCS-23-003, University of Utah, November 2023.

[8] Thomas C. Henderson, Amar Mitiche, Xiuyi Fan, and David Sacharny. Some Explorations in SAT. Technical Report UUCS-21-016, University of Utah, July 2021.

[9] Thomas C. Henderson, David Sacharny, Amar Mitiche, Xiuyi Fan, Amelia Lessen, Ishaan Rajan, and Tessa Nishida. Chop-SAT: A New Approach to Solving SAT and Probabilistic SAT for Agent Knowledge Bases. In *International Conference on Agents and Artificial Intelligence*, Lisbon, Spain, February 2023.

[10] M. Sipser. *Introduction to the Theory of Computation*. Cengage Learning, Independence, KY, 2012.