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ON THE COMPLEXITY OF FOUR POLYHEDRAL SET CONTAINMENT PROBLEMS

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A nonempty closed convex polyhedron X can be represented either as $X=\{x\colon Ax\leqslant b\}$, where (A, b) are given, in which case X is called an H-cell, or in the form $X=\{x\colon x=U\lambda+V\mu, \sum\lambda_j=1, \lambda\geqslant 0, \mu\geqslant 0\}$, where (U, V) are given, in which case X is called a W-cell. This note discusses the computational complexity of certain set containment problems. The problems of determining if $X\not\subseteq Y$, where (i) X is an H-cell and Y is a closed solid ball, (ii) X is an H-cell and Y is a W-cell, or (iii) X is a closed solid ball and Y is a W-cell, are all shown to be NP-complete, essentially verifying a conjecture of Eaves and Freund. Furthermore, the problem of determining whether there exists an integer point in a W-cell is shown to be NP-complete, demonstrating that regardless of the representation of X as an H-cell or W-cell, this integer containment problem is NP-complete.

Key words: Computational Complexity, NP-Complete, Linear Program, Polyhedron, Cell.

1. Introduction and preliminaries

A nonempty closed convex polyhedron X can be represented either in the form $X=\{x\colon Ax\leqslant b\}$, where (A, b) are given, in which case X is called an *H-cell* (H for halfspaces), or in the form $X=\{x\colon x=U\lambda+V\mu, \sum\lambda_j=1, \lambda\geqslant 0, \mu\geqslant 0\}$ where (U, V) are given, in which case X is called a *W-cell* (W for weighting of points). When X is represented as a W-cell, the columns of U and V contain the extreme points and extreme rays of X, respectively. The computational complexity of many problems related to polyhedra depend on the polyhedral representation as an H-cell or a W-cell. For example, consider a linear program, which can be stated as

$$\text{maximize } c^t x \quad \text{subject to } x\in X,$$

where X is a polyhedron. If X is an H-cell, this is the usual linear program, whose solution time, while polynomial, is by no means negligible. However, if X is represented as a W-cell, the linear programming problem becomes trivial. As another example, consider the problem of testing if $\bar{x}\in X$ for a given $\bar{x}$, where X is a polyhedron. If X is an H-cell, the problem is trivial, whereas if X is a W-cell, the problem reduces to solving a linear program.

This note discusses the complexity of two types of problems. The first problem is the *set containment problem* (SCP), that of determining if $X\subseteq Y$, where X (resp. Y) is a *cell*, defined to be either a polyhedron (an H-cell or a W-cell), or a closed solid ball of the form $\{x: (x-c)^t(x-c) \leq r^2\}$, in which case X (resp. Y) is called a *B-cell*. There are nine forms of SCP corresponding to X and Y each being given as an H-cell, W-cell, or B-cell. For notational convenience, a particular form of SCP will be denoted, e.g., by (W, B), where X is a W-cell and Y is a B-cell. In Eaves and Freund [1], SCP is shown to be solvable as a linear program for the six forms (HH), (WH), (BH), (WW), (WB), and (BB), thus showing that these problems are solvable in polynomial time. Eaves and Freund also conjectured that the forms (HW), (BW), and (HB) are 'intractable'. In Section 2 of this article, we show that these three forms of SCP are co-NP-complete, (i.e., that the corresponding *non*containment problems are NP-complete), thus essentially confirming the conjecture.

Section 3 addresses the computational complexity of the *integer containment problem* (ICP), that of finding an integer point in a given polyhedron X in the case that X is a W-cell. Karp [4] showed that when X is an H-cell, the corresponding ICP is NP-complete. Herein, it is shown that ICP is also NP-complete when X is a W-cell.

The notation used is standard. Let $\mathbb{R}^n$ be n-dimensional Euclidean space. The Euclidean norm of $x \in \mathbb{R}^n$ is represented by $\|x\|$. Let $e = (1, 1, 1, \ldots, 1)$ where the dimension is clear from the context. Let $Q^{m\times n}$, Q^n be the set of rational $m \times n$ matrices and n-vectors, respectively. Define

$$\{a, b\}^n = \{x \in \mathbb{R}^n: x_j = a \text{ or } b, j = 1, \ldots, n\}.$$

2. Three NP-complete cases of the set containment problem

The three set containment problems of interest, forms (HB), (HW), and (BW), can be stated in their noncontainment form, as:

(HB) *Given*: $(A, b, c, r^2) \in (Q^{m\times n}, Q^m, Q^n, Q^1)$.

Question: Is $X \not\subseteq Y$, where $X = \{x \in \mathbb{R}^n: Ax \leq b\}$ and $Y = \{x \in \mathbb{R}^n: (x-c)^t(x-c) \leq r^2\}$?

(HW) *Given*: $(A, b, U, V) \in (Q^{m\times n}, Q^m, Q^{n\times k}, Q^{n\times p})$.

Question: Is $X \not\subseteq Y$, where $X = \{x \in \mathbb{R}^n: Ax \leq b\}$ and $Y = \{x \in \mathbb{R}^n: x = U\lambda + V\mu, e^t\lambda = 1, \lambda \geq 0, \mu \geq 0\}$?

(BW) *Given*: $(c, r^2, U, V) \in (Q^n, Q^1, Q^{n\times k}, Q^{n\times p})$.

Question: Is $X \not\subseteq Y$, where $X = \{x \in \mathbb{R}^n: (x-c)^t(x-c) \leq r^2\}$ and $Y = \{x \in \mathbb{R}^n: x = U\lambda + V\mu, e^t\lambda = 1, \lambda \geq 0, \mu \geq 0\}$?

Note that problems (HB), (HW), and (BW) are elements of NP. For a given instance of (HB) or (HW), the resolution of $X \not\subseteq Y$ can be accomplished by determining an extreme point or extreme ray $\bar{x}$ of X that is not an element or ray of Y, respectively; the size of $\bar{x}$ is polynomially bounded in the size of the input data (see, e.g., Gantmacher [2]) and so HB and HW are elements of NP. For a given instance of (BW), suppose that $X \not\subseteq Y$. Then, either $c \notin Y$ or $c \in Y$ and there is an $(n-1)$-face F of Y such that the shortest Euclidean distance from c to the hyperplane Z containing F is less than r. If the former is true, the test $c \notin Y$ amounts to solving a linear program, which is polynomially bounded in the size of the input data. If the latter is true, there exists a submatrix U' consisting of columns of U which are extreme points of F, and submatrix V' consisting of columns of V which are extreme rays of F, such that the hyperplane Z containing F is determined by a unique (up to positive multiple) solution $(\bar{\pi}, \bar{\alpha})$ to $\pi U' = \alpha e$, $\pi V' = 0$, $\pi U \leq \alpha e$, $\pi V \leq 0$, $\pi \neq 0$, where $Z = \{x \mid \bar{\pi} \cdot x = \bar{\alpha}\}$. The size of a solution $(\bar{\pi}, \bar{\alpha})$ to the above system can be polynomially bounded in the data (U, V) and the shortest Euclidean distance from c to Z is given by $(\bar{\alpha} - \bar{\pi} \cdot c)/\sqrt{\bar{\pi} \cdot \bar{\pi}}$. The test that $(\bar{\alpha} - \bar{\pi} \cdot c)^2/\bar{\pi} \cdot \bar{\pi} < r^2$ is also polynomially bounded in the data $(\bar{\alpha}, \bar{\pi}, c, r^2)$ and so problem (BW) is in the class NP.

Consider the following version of the integer containment problem:

(ICP1) *Given*: $\bar{A} \in Q^{m \times n}$.

Question: Is there a $\pi \in \{-1, 1\}^n$ that satisfies $\bar{A}\pi \leq e$?

This classical integer linear inequalities problem is NP-complete, even if m is restricted to be 2, as there is an elementary transformation from the number partition problem. In order to prove that our three cases of SCP are NP-complete, we will demonstrate a transformation of ICP1 to our desired problem.

Our main result in this section is the following:

Theorem 1. *The set containment problems* (HB), (HW), *and* (BW) *are NP-complete.*

Before proceeding to the proofs, we define a few more terms and we state an elementary property concerning linear programs defined over the rationals.

For each matrix $\bar{A}$, let $P(\bar{A}) = \{x\colon \bar{A}x \leq e, -e \leq x \leq e\}$. Thus the integer containment problem ICP1 can be stated as follows: Does $P(\bar{A}) \cap \{-1, 1\}^n \neq \emptyset$?

For a given rational matrix A, we will let max (A) denote the maximum absolute value of a numerator or denominator of a component of A; e.g., $\max(\frac{2}{11}, -\frac{14}{2})) = 14$. (The numerator and divisor can have a common divisor.)

For two sets S, T, let $d(S, T)$ be the infimum of the distance between the two sets, where the supremum norm is used. In the proofs, we will use the following elementary lemma.

Lemma 1. *If* $P(\bar{A}) \cap \{-1, 1\}^n = \emptyset$, *then* $d(P(\bar{A}), \{-1, 1\}^n) > (2\max(\bar{A})^{(n^3+n)} n!)^{-1}$.

Proof. Let $z^* = d(P(\bar{A}), \{-1, 1\}^n)$, and $z^*(y) = d(P(\bar{A}), \{y\})$; then

$$z^* = \min(z^*(y)\colon y \in \{-1, 1\}^n),$$

and

$$z^*(y) = \text{minimum } z,$$
$$\text{subject to } z + (x_j - y_j) \geq 0, \quad j = 1, \ldots, n,$$
$$z - (x_j - y_j) \geq 0, \quad j = 1, \ldots, n,$$
$$x \in P(\bar{A}).$$

We now claim that $z^*(y) > (2 \max(\bar{A})^{(n^3+n)}(n!))^{-1}$ for any $y \in \{-1, 1\}^n$. To see this, let $(x^*, \bar{z})$ be a point in $P(\bar{A})$ of minimum distance to y, and without loss of generality we may assume that $(x^*, \bar{z})$ is an extreme point of the feasible region of the above linear program. Therefore $(x^*, \bar{z}) = B^{-1}f$ where B is a row basis of the linear program and f is a vector of 0's and 1's of the right-hand side components corresponding to B. B can be written as $B = d^{-1}C$ where d is a common denominator of B, and C is an integral matrix. Because $B^{-1} = dC^{-1} = d(\text{adj}(C)/\det(C))$, a denominator for B^{-1} is $\det(C)$. Because $d \leq \max(\bar{A})^{n^2}$ and $\max(C) \leq (\max(\bar{A})^{n^2+1})$, we obtain

$$\det(C) \leq \max(C)^n n! \leq (\max(\bar{A})^{n^2+1})^n n!$$
$$= \max(\bar{A})^{(n^3+n)} n! < 2 \max(\bar{A})^{(n^3+n)} n!.$$

Because the numerator of $\bar{z}$ is a positive integer, we have $\bar{z} \geq (\det(C))^{-1}$, and so the above bound on $\det(C)$ provides a bound for $\bar{z}$ and hence z^*. □

Henceforth, for each $A \in Q^{m \times n}$, let $M(A) = (2\max(\bar{A})^{(n^3+n)} n!)$. Note that the size of $M(A)$ is $O(n^3 \log(1 + \max(A))$, which is polynomial in the size of A.

Proof that (HB) is NP-complete. Let $\bar{A}$ be an instance of ICP1, and let $\varepsilon = [M(\bar{A})]^{-1}$. Let $X = P(\bar{A})$ and let $Y = \{y \in \mathbb{R}^n: y^t y \leq n - \varepsilon\}$. Consider the instance of (HB) of determining if $X \not\subseteq Y$.

Suppose first that $X \subseteq Y$. Then $\|x\|^2 \leq n - \varepsilon < n$ for any $x \in P(\bar{A})$ and thus $P(\bar{A}) \cap \{-1, 1\}^n = \emptyset$.

Conversely, suppose that $X \not\subseteq Y$. Let $x \in P(\bar{A})$ be selected so that $x \notin Y$. Since $-e \leq x \leq e$ and $x^t x \geq n - \varepsilon$, it follows that $|x_j| \geq 1 - \varepsilon$ for each j and thus $d(x, \{-1, 1\}^n) \leq \varepsilon$. It follows that $d(P(\bar{A}), \{-1, 1\}^n) \leq \varepsilon$, and thus by Lemma 1, we conclude that $P(\bar{A}) \cap \{-1, 1\}^n \neq \emptyset$. □

Proof that (HW) is NP-complete. Let $\bar{A}$ be an instance of ICP1, and let $\varepsilon = [M(\bar{A})]^{-1}$. Let $X = P(\bar{A})$, and let $Y = \{y: \sum_{j=1}^n |y_j| \leq n - \varepsilon\}$. Note that Y may be polynomially represented as the W-cell $\{y: y = U\lambda, \lambda \geq 0, e^t\lambda = 1\}$ by letting $U = [(n - \varepsilon)I, (n - \varepsilon)(-I)]$. Now consider the instance (H, W) of determining if $X \not\subseteq Y$.

Suppose first that $X \subseteq Y$. Then any $x \in P(\bar{A})$ must satisfy $\sum_j |x_j| \leq n - \varepsilon$ and thus $P(\bar{A}) \cap \{-1, 1\}^n = \emptyset$.

Suppose next that $X \not\subseteq Y$. Let $x \in X$ be chosen so that $\sum_j |x_j| > n - \varepsilon$. Since $-e \leq x \leq e$, it follows that $1 - \varepsilon \leq |x_j| \leq 1$ for each $j = 1, \ldots, n$ and thus $d(x, \{-1, 1\}^n) \leq \varepsilon$.

Therefore $d(P(\bar{A}), \{-1,1\}^n) \leq \varepsilon$, and thus by Lemma 1 we conclude that $P(\bar{A}) \cap \{-1,1\}^n \neq \emptyset$. $\square$

Proof that (BW) is NP-complete. Let $\bar{A}$ be an instance of ICP1 and let $\varepsilon = M(\bar{A})^{-1}$. Let $X = \{x \in \mathbb{R}^n: \|x\|^2 \leq 1/(n-\varepsilon)\}$ and let $Y = \{y \in \mathbb{R}^n: \pi^t y \leq 1 \text{ for all } \pi \in P(\bar{A})\}$. We first show that Y can be represented as the W-cell $Y' = \{y: y = \lambda^1 - \lambda^2 + \bar{A}^t\lambda^3, \lambda^1, \lambda^2, \lambda^3 \geq 0, e^t\lambda^1 + e^t\lambda^2 + e^t\lambda^3 = 1\}$. It is easy to see that $Y' \subseteq Y$ by premultiplying any $y \in Y'$ by $\pi \in P(\bar{A})$. To show that $Y \subseteq Y'$, suppose that $\bar{y} \in Y$. Then $\pi^t\bar{y} \leq 1$ for any $\bar{y} \in P(\bar{A})$. If $y \notin Y'$, the linear system $\lambda^1 - \lambda^2 + \bar{A}^t\lambda^3 = \bar{y}$, $e^t\lambda^1 + e^t\lambda^2 + e^t\lambda^3 = 1$, $\lambda^1 \geq 0$, $\lambda^2 \geq 0$, $\lambda^3 \geq 0$ has no solution. In this case, by a theorem of the alternative, there exists $\pi \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, such that $\pi - \mu e \leq 0$, $-\pi - \mu e \leq 0$, $\bar{A}\pi - \mu e \leq 0$, and $\pi^t\bar{y} - \mu > 0$. It is simply to verify that we must have $\mu > 0$, and thus we can assume $\mu = 1$, whereby $\pi \in P(\bar{A})$ and $\pi^t\bar{y} > 1$, i.e. $\bar{y} \notin Y$, a contradiction. Thus $Y' = Y$.

Consider the case of (BW) of determining if $X \subseteq Y$. Suppose first that $P(\bar{A}) \cap \{-1,1\}^n \neq \emptyset$, and let $v \in P(\bar{A}) \cap \{-1,1\}^n$. Let $\bar{v} = (n-\varepsilon)^{-1/2}\|v\|^{-1}v$. Note first that $\bar{v}^t\bar{v} = (n-\varepsilon)^{-1}$ and so $\bar{v} \in X$. Also note that $v^t\bar{v} = (n-\varepsilon)^{-1/2}\|v\| > 1$, and so $\bar{v} \notin Y$. We conclude in this case that $X \nsubseteq Y$. Thus if $X \subseteq Y$, $P(\bar{A}) \cap \{-1,1\}^n = \emptyset$.

Next consider the case that $X \nsubseteq Y$. In this case there exists $x \in X$ and $\pi \in P(\bar{A})$ such that $\pi^t x > 1$. Moreover the value of $\bar{x} \in X$ which maximizes $\pi^t\bar{x}$ is uniquely given by $\bar{x} = (n-\varepsilon)^{-1/2}\|\pi\|^{-1}\pi$ whenever $\pi \neq 0$. Thus we may assume without loss of generality that $x = (n-\varepsilon)^{-1/2}\|\pi\|^{-1}\pi$. It follows that $\|\pi\|^2 = \pi^t\pi = (n-\varepsilon)^{1/2}\|\pi\|\pi^t x > (n-\varepsilon)^{1/2}\|\pi\|$, and thus $\|\pi\|^2 > n - \varepsilon$. Since $-e \leq \pi \leq e$, $1 - \varepsilon \leq |\pi_j| \leq 1$ for $j = 1, \ldots, n$ and thus $d(\pi, \{-1,1\}^n) \leq \varepsilon$. We conclude that $d(P(\bar{A}), \{-1,1\}^n) \leq \varepsilon$ and thus by Lemma 1, $P(\bar{A}) \cap \{-1,1\}^n \neq \emptyset$. $\square$

3. The complexity of finidng an integer element of a polyhedron

It is well known (see for example Garey and Johnson [3]) that the problem of determining whether there is an integer point in an H-cell is NP-complete. In this section we show an analogous result for integral containment in a W-cell. Consider

ICP2: *Given*: $(\bar{U} \in Q^{n \times k})$.

Question: Is there an integral n-vector $\pi \in X$, where

$$X = \{x \in \mathbb{R}^n: x = \bar{U}\lambda, e^t\lambda = 1, \lambda \geq 0\}?$$

Theorem 2. *The problem* ICP2 *is NP-complete*

Proof. Note first that ICP2 is an element of NP since if $\pi \in X$ is integral, then the size of π is polynomially bounded in the size of $\bar{U}$, and we can demonstrate that $\pi \in X$ by solving a linear program in polynomial time.

To show that ICP2 is NP-complete, we carry out a transformation from the following 0–1 knapsack problem.

Input: Integers $a_1, \dots, a_n, b$.
Question: Is there a vector $y \in \{0, 1\}^n$ such that $\sum_{i=1}^n a_i y_i = b$?

The above problem is known to be NP-complete (see for example Garey and Johnson [3]).

Suppose that $a_1, \dots, a_n, b$ is an instance of the above knapsack problem. We transform this instance into a problem in modular arithmetic as follows: Are there vectors λ, s satisfying:

$$\left(\sum_{j=1}^{n} a_j\lambda_j - b\lambda_{n+1}\right) \quad \text{is integral,} \tag{1a}$$

$$(n+1)\lambda_j \quad \text{is integral for } j = 1, \dots, n+1, \tag{1b}$$

$$(2n)^{-1}(\lambda_j + s_j - \lambda_{n+1}) \quad \text{is integral for } j = 1, \dots, n, \tag{1c}$$

$$s_1 + \cdots + s_n + \lambda_1 + \cdots + \lambda_{n+1} = 1, \tag{1d}$$

$$s, \lambda \geqslant 0. \tag{1e}$$

First note that (1a)–(1e) is a special case of ICP2 in which U has $2n+1$ columns each of which is in Q^{2n+2}.

We claim that there is a feasible solution to system (1) if and only if there is a solution to the knapsack problem.

Suppose first that $y \in \{0, 1\}^n$ is feasible for the knapsack problem. Let $\lambda_j = y_j/(n+1)$ for $j = 1, \dots, n$ and let $s_j = 1/(n+1) - \lambda_j$. Finally, let $\lambda_{n+1} = 1/(n+1)$. It is easy to verify that λ, s satisfy (1).

Suppose next that λ, s satisfy (1). If we subtract each of the n constraints of (1c) from $(2n)^{-1}$ of constraint (1d), we obtain the constraint

$$((n+1)/2n)\lambda_{n+1} - 1/2n \text{ is integral.} \tag{1f}$$

Since $0 \leqslant \lambda_{n+1} \leqslant 1$, we conclude from (1f) that

$$\lambda_{n+1} = 1/(n+1). \tag{1g}$$

We conclude from (1g), (1c) and (1d) that

$$\lambda_j + s_j = 1/(n+1) \quad \text{for } j = 1, \dots, n \tag{1h}$$

and by (1h) and (1b) we conclude that

$$\lambda_j = 0 \text{ or } 1/(n+1) \quad \text{for } j = 1, \dots, n. \tag{1i}$$

From (1g), (1i) and (1a) we conclude that $y = (y_1, \dots, y_n)$ is feasible for the knapsack problem, where $y_j = (n+1)\lambda_j$, $j = 1, \dots, n$, completing the proof. □

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