Introduction to Interior Point Methods

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These slides do not contain all the topics intended for discussion ..... Watch out errors are everywhere!
In the meantime, I am happy to receive your suggestions, corrections and comments.

But, "I won't leave any unfinished manuscripts" Harold Robbins - American author with 25 bestsellers.

Topics

- Basic Principles of the Interior Point (Barrier) Methods
- Primal-Dual Interior Point methods
- Primal-Dual Interior Point methods for Linear and Quadratic Optimization
- Primal-Dual-Interior Point methods for Nonlinear Optimization
- Current Issues
- Conclusion
- References and Resources

Basics of the Interior Point Method

Consider

$$(NLP) \qquad \min_x f(x)$$
$$s.t.$$
$$g_i(x) \geq 0, i = 1, 2, \ldots, m_1;$$
$$h_j(x) = 0, j = 1, 2, \ldots, m_2;$$
$$x \geq 0,$$

where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ are at least once differentiable functions, $x_{min}, x_{max} \in \mathbb{R}^n$ are given vectors.

Feasible set of NLP:

$$\mathcal{F} := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m_1; \; h_j(x) = 0, j = 1, \ldots, m_2; x \geq 0\}.$$

Basics of the Interior Point Method...

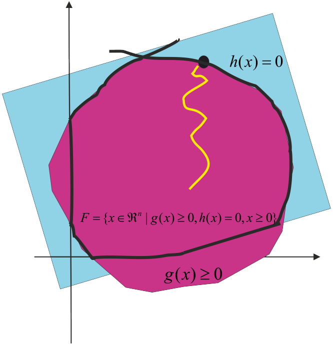

Figure: Feasible set $\mathcal{F}$

Idea of the interior point method:

• to iteratively approach the optimal solution from the *interior of the feasible set*

Basics of the Interior Point Method...

Therefore (**requirements for IPM**):

- the *interior of the feasible set* should not be empty
- almost all iterates should remain in (the interior of the) feasible set

Question:

When is the interior of the feasible set non-empty?

Answer:

(i) if there is $\overline{x} \in \mathbb{R}^n$ such that

$$g_i(\overline{x}) > 0, i = 1, \ldots, m_1; h_j(\overline{x}) = 0, j = 1, \ldots, m_2; \overline{x} > 0.$$

(ii) if the Mangasarian-Frmomovitz Constraint Qualification (MFCQ) is satisfied at a feasible point $\overline{x}$,

then **the interior of the feasible set of NLP is non-empty**.

What is MFCQ ?

Let $\overline{x} \in \mathcal{F}$; i.e. $\overline{x}$ is a feasible point of NLP.

Active constraints

- An inequality constraint $g_i(x)$ is said to be active at $\overline{x} \in \mathcal{F}$ if

$$g_i(\overline{x}) = 0.$$

- The set

$$\mathcal{A}(\overline{x}) = \{i \in \{1, \ldots, m_1\} \mid g_i(\overline{x}) = 0\}$$

index set of active inequality constraints at $\overline{x}$.

$$(NLP) \quad \min_x \{f(x) = x_1^2 - x_2^2\} \quad s.t. \qquad \begin{aligned} g_1(x) &= x_1^2 + x_2^2 + x_3^2 + 3 \geq 0, \\ g_2(x) &= 2x_1 - 4x_2 + x_3^2 + 1 \geq 0, \\ g_3(x) &= -5x_1 + 3x_2 + 2 \geq 0, \\ x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

What is MFCQ ?...

The vector $\overline{x}^\top = (1, 1, 1)$ is feasible to the NLP and

$$g_2(\overline{x}) = 0 \text{ and } g_3(\overline{x}) = 0,$$

the active index set is $\mathcal{A}(\overline{x}) = \{2, 3\}$.

Mangasarian-Fromowitz Constraint Qualification

Let $\overline{x} \in \mathcal{F}$ (feasible point of NLP). Them MFCQ is said to be satisfied at $\overline{x}$ if there is a vector $d \in \mathbb{R}^n, d \neq 0$, such that (i)

(i) $\quad d^\top \nabla g_i(\overline{x}) > 0, i \in \mathcal{A}(\overline{x}),$ and

(ii) $\quad d^\top \nabla h_1(\overline{x}) = 0, d^\top \nabla h_2(\overline{x}) = \ldots, d^\top \nabla h_{m_2}(\overline{x}) = 0.$

What is MFCQ ?...

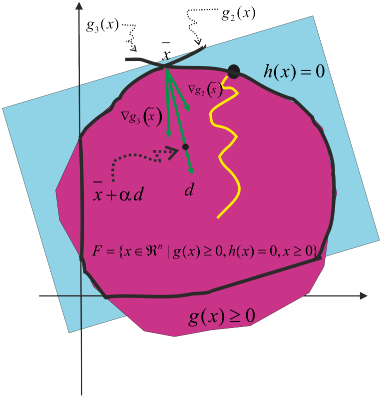

Figure: A Mangasarian-Fromowitz Vector d

- d forms an acute angle ($< 90^0$) with each $\nabla g_i(\overline{x}), i \in \mathcal{A}(\overline{x})$.

What is MFCQ ?...

An implications of the MFCQ:

There is α such that

- $\bar{x} + \alpha d > 0$.
- $g(\bar{x} + \alpha d) \approx g(\bar{x}) + \alpha d^\top \nabla g_i(\bar{x}) > 0, i = 1, \ldots, m_1$,
- $h_j(\bar{x} + \alpha d) \approx h_j(\bar{x}) + \alpha d^\top \nabla h_j(\bar{x}) = 0, j = 1, \ldots, m_2$.

$\Rightarrow$ $\bar{x} + \alpha d$ **is in the interior of the feasible set** $\mathcal{F}$.

$\Rightarrow$ **The interior of the feasible set is not empty**.

Example (continued...)

- $\nabla g_2(\bar{x}) = (2, -4, 2)$ and $\nabla g_3(\bar{x}) = (-5, 3, 0)$.
- for $d^\top = (-1, 0, 2)$ we have $d^\top \nabla g_2(\bar{x}) > 0$ and $d^\top \nabla g_3(\bar{x}) > 0$; and
- $x = (1, 1, 1) + \underbrace{\frac{1}{10}}_{=\alpha} (-1, 0, 2) > 0$.

MFCQ guarantees that the interior of $\mathcal{F}$ is not empty.

Forcing iterates remain in the interior of $\mathcal{F}$

Question:

How to force almost all iterates remain in the interior of the feasible set $\mathcal{F}$?

Answer:

Use barrier functions?

A well-known barrier function is the **logarithmic barrier function**

$$\mathcal{B}(x, \mu) = f(x) - \mu \left(\sum_{i=1}^{m_1} \log(g_i(x)) + \sum_{l=1}^{n} \log(x_l) \right)$$

where μ is known as **barrier parameter**.

- The logarithmic terms $\log(g_i(x))$ and $\log(x_l)$ are defined at points x for which $g_i(x) > 0$ and $x_l > 0, l = 1, \dots, n$.

Basics of the Interior Point Method...

- Instead of the problem NLP, consider the parametric problem

$$(NLP)_\mu \qquad \min_x \mathcal{B}(x,\mu)$$
$$s.t.$$
$$h_j(x) = 0, j = 1, \ldots, m_2.$$

- To find an optimal solution x_μ of $(\text{NLP})_\mu$ for a fixed value of the barrier parameter μ.

Lagrange function of $(\text{NLP})_\mu$:

$$\mathcal{L}_\mu(x,\lambda) = f(x) - \mu\left(\sum_{i=1}^{m_1}\log(g_i(x)) + \sum_{l=1}^{n}\log(x_l)\right) - \sum_{j=1}^{m_2}\lambda_j h_j(x).$$

Basics of the Interior Point Method...

Necessary optimality (Karush-Kuhn-Tucker) condition:

for a given μ, a vector x_μ is a minimum point of $(\text{NLP})_\mu$ if there is a Lagrange parameter λ_μ such that, the pair (x_μ, λ_μ) satisfies:

$$\begin{aligned} \nabla_\lambda \mathcal{L}_\mu(x,\lambda) &= 0 \\ \nabla_x \mathcal{L}_\mu(x,\lambda) &= 0 \end{aligned}$$

$\Rightarrow$ Thus we need to solve the system

$$\begin{aligned} -h(x) &= 0 \\ \nabla f(x) - \mu \left(\sum_{i=1}^{m_1} \frac{1}{g_i(x)} \nabla g_i(x) + \sum_{l=1}^{m_1} \frac{1}{x_l} e_l \right) + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x) &= 0 \end{aligned}$$

- Commonly, this system is solved iteratively using the Newton Method.

Basics of the Interior Point Method...

Newton method to solve the system of nonlinear equations $F_\mu(x,\lambda) = 0$ for a fixed μ, where

$$F_\mu(x,\lambda) = \begin{pmatrix} h(x) \\ \nabla f(x) - \mu\left(\sum_{i=1}^{m_1} \frac{1}{g_i(x)} \nabla g_i(x) + \sum_{l=1}^{m_1} \frac{1}{x_l} e_l\right) + \\ + \sum_{j=1}^{m_2} \lambda_j \nabla h_j(x) \end{pmatrix}$$

Algorithm:

Step 0: Choose (x_0, λ_0).

Step k: • Find $(\Delta_x^k, \Delta_\lambda^k) = d$ by solving the linear system

$$\mathbf{J}_{\mathbf{F}_\mu}(\mathbf{x_k}, \lambda_\mathbf{k})\mathbf{d} = -\mathbf{F}_\mu(\mathbf{x_k}, \lambda_\mathbf{k})$$

- Determine a step length α_k
- Set $x_{k+1} = x_k + \alpha_k \Delta_x^k$ and $\lambda_{k+1} = x_k + \alpha_k \Delta_\lambda^k$

STOP if convergence is achieved; otherwise CONTINUE.

Basics of the Interior Point Method...

- For each give μ, the above algorithm can provide a minimal point x_μ of the problem $(\text{NLP})_\mu$.

Question: What is the relation between the problem NLP and $(\text{NLP})_\mu$?

Question: How to choose μ's?

Answer(a general strategy): choose a sequence $\{\mu_k\}$ of decreasing, sufficiently small non-negative barrier parameter values

- to obtain associated sequence $\{x_{\mu_k}\}$ optimal solutions of $(NLP)_{\mu_k}$.

Properties

- The optimal solutions x_μ lie in the interior of the feasible set of NLP.
- The solutions x_{μ_k} converge to a solution x^* of NLP; i.e.

$$\lim_{\mu \searrow 0^+} x_\mu = x^*.$$

Drawbacks of the primal barrier interior

$$J_{F_\mu}(x,\lambda) = \begin{pmatrix} H(x) - \mu \underbrace{\left(\sum_{i=1}^{m_1} \frac{1}{g_i(x)} \left[\nabla g_i(x) \nabla g_i(x)^\top + G_i(x) \right] - \sum_{l=1}^{m_1} \frac{1}{x_l^2} e_l \right)}_{:=D(x)} + \sum_{j=1}^{m_2} \lambda_j \nabla \mathcal{H}_j(x) & J_h(x) \\ [J_h(x)]^\top & 0 \end{pmatrix},$$

where, $H(x)$ is the Hessian matrix of $f(x)$, $J_h(x)$ is the Jacobian matrix of $h(x)^\top = (h_1(x), h_2(x), \ldots, h_{m_2}(x))$, $G_i(x)$ is the Hessian matrix of $g_i(x)$, $\mathcal{H}_j(x)$ is the Hessian matrix of $h_j(x)$.

Drawback: as the values of μ get closer to 0 the matrix D can become ill-conditioned.

Example (continued):

For our example we have

$$D(x) = \frac{1}{g_1(x)} \begin{bmatrix} 4x_1^2 + 2 & 4x_1x_2 & 4x_1x_3 \\ 2x_1x_2 & 4x_1^2 + 2 & 2x_1x_2 \\ 4x_1x_3 & 4x_3x_2 & 4x_3^2 + 2 \end{bmatrix} + \frac{1}{g_2(x)} \begin{bmatrix} 4 & -8 & 4x_3 \\ -8 & 16 & -8x_3 \\ 4x_3 & -8x_3 & 4x_3 + 2 \end{bmatrix} + \frac{1}{g_3(x)} \begin{bmatrix} 25 & -15 & 0 \\ -15 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} - X^{-2}$$

where $X = diag(x)$. For example, at the feasible interior point $x^\top = (1, 2, 8)$ we have $cond(D) \approx 113.6392$, which is large.

Drawbacks of the primal barrier interior

Note that:

• the matrix $\nabla g(x) [\nabla g(x)]^\top$ is of rank 1, so not invertible and has large condition number.

• the expression $\frac{1}{g(x)}$ gets larger as $g(x)$ gets smaller, usually near to the boundary of the feasible region.

Advise: Do not use the constraint function $g_i(x) \geq 0, i = 1, \ldots, m_1$ directly with the logarithmic barrier function .

Instead, introduce slack variables $s = (s_1, s_2, \ldots, s_{m_1})$ for inequality constraints so that:

$$g_i(x) - s_i = 0, s_i \geq 0, i = 1, \ldots, m_1.$$

(That is, we lift the problem into a higher dimension by adding new variables, so that we have to work with $z = (x, s) \in \mathbb{R}^{n+m_1}$. Frequently, in higher dimensions, we may have a better point of view.)

The Primal-Dual Interior Point Method

This leads to the problem

$$(NLP)_\mu \qquad \min_{(x,s)} \left\{ f(x) - \mu \left(\sum_{l=1}^{n} log(x_l) + \sum_{i=1}^{m_1} log(s_i) \right) \right\}$$

$$s.t.$$

$$g_i(x) - s_i = 0, i = 1, \ldots, m_1$$

$$h_j(x) = 0, j = 1, \ldots, m_2.$$

only with equality constraints and objective function with barrier terms on the variables.

$$(NLP)_\mu \qquad \min_{(x,s)} \left\{ f(x) = \left(x_1^2 - x_2^2\right) - \mu \left[\sum_{i=1}^{3} (\log s_i + \log x_i) \right] \right\} \quad (1)$$

$$s.t. \quad (2)$$

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 + 3 - s_1 = 0,$$

$$g_2(x) = 2x_1 - 4x_2 + x_3^2 + 1 - s_2 = 0,$$

$$g_3(x) = -5x_1 + 3x_2 + 2 - s_3 = 0. \quad (3)$$

Primal-dual Interior Method for LOPs

- Consider a standard linear optimization problem

$$(LOP) \qquad \min_x c^\top x \quad s.t. \quad Ax = b, \quad x \geq 0$$

where A is $m \times n$ matrix, $b \in \mathbb{R}^n$.

- The **dual problem** to LOP is:

$$(LOP)_D \qquad \max_{(\lambda,s)} b^\top \lambda \quad s.t. \quad A^\top \lambda + s = c.$$

Here, s is slack variable.

Primal-dual Interior Method for LOPs

The Lagrange function of LOP:

$$\mathcal{L}(x, \lambda, s) = c^\top x - \lambda^\top (Ax - b) - \sum_{i=1}^{m} s_i x_i,$$

where:

- $\lambda^\top = (\lambda_1, \ldots, \lambda_m)$ is a vector of Lagrange multipliers associated with the equality constraints $Ax = b$,and
- $s = (s_1, \ldots, s_n)$ is a vector of Lagrange-multipliers associated with $x \geq 0$; hence $s \geq 0$.

- Here, the Lagrange-multiplier vector s is same as the slack variable s in the dual problem $(\text{LOP})_D$.

Primal-dual Interior Method for LOPs...

- The optimality criteria for x^* to be a solution of the primal problem (P) and (λ^*, s^*) to be a solution of dual problem (D) is that (x^*, λ^*, s^*) should satisfy:

$$
\begin{aligned}
c - A^\top \lambda - s &= 0 && (4)\\
Ax &= b && (5)\\
XS\mathbf{e} &= 0 && (6)\\
(x, s) &\geq 0 \quad . && (7)
\end{aligned}
$$

where:

$$X = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}, S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}, \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Primal-dual Interior Method ...

Question:

Where is the relation with the interior point method?

- The barrier function associated to LOP is

$$\mathcal{B}(x,\mu) = f(x) - \mu \sum_{i=1}^{m_1} \log(x_i)$$

- The barrier problem will be

$$(NLP)_\mu \qquad \min_x \left\{ f(x) - \mu \sum_{i=1}^{m_1} \log(x_i) \right\}$$

$$s.t.$$

$$Ax = b.$$

- The Lagrange function of the barrier Problem

$$\mathcal{L}_\mu(x,\lambda) = c^\top x - \lambda^\top (Ax - b) - \mu \sum_{i=1}^{n} \log(x_i).$$

Primal-dual Interior Method for LOPs...

- For a given μ, the pair (x_μ, λ_μ) is a solution of the primal problem NLP_μ if it satisfies the optimality conditions:

$$\nabla_x \mathcal{L}_\mu(x,\lambda) = 0 \quad (8)$$
$$\nabla_\lambda \mathcal{L}_\mu(x,\lambda) = 0 \quad (9)$$
$$x > 0. \quad (10)$$

$\Rightarrow$ **KKT Conditions**

$$c - A^\top\lambda - \underbrace{\mu X^{-1}\mathbf{e}}_{:=s} = 0,$$
$$Ax = b,$$
$$x > 0.$$

$\Rightarrow$ **KKT Conditions**

$$c - A^\top\lambda - s = 0,$$
$$Ax = b,$$
$$s = \mu X^{-1}\mathbf{e}$$
$$(x,s) > 0.$$

- Where : $s = \mu X^{-1}\mathbf{e}$.

Primal-dual Interior Method for LOPs...

- It follows (since $x_i \neq 0$) that $s_i = \frac{\mu}{x_i} > 0 \Rightarrow s_i x_i = \mu, i = 1, \ldots, n$.

$$\begin{bmatrix} s_1 x_1 & & & \\ & s_2 x_2 & & \\ & & \ddots & \\ & & & s_n x_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\Rightarrow$

$$\underbrace{\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}}_{=X} \underbrace{\begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_n \end{bmatrix}}_{=S} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{=\mathbf{e}} = \mu \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{=\mathbf{e}}$$

$\Rightarrow XS\mathbf{e} = \mu\mathbf{e}$.

Primal-dual Interior Method for LOPs...

• Now, the optimality conditions, for the barrier problem NLP_{μ}, given in (8) - (10) can be equivalently as:

$$
\begin{aligned}
Ax &= b, && (11)\\
A^{\top}\lambda + s &= c, && (12)\\
XS\mathbf{e} &= \mu\mathbf{e} && (13)\\
(x, s) &> 0. && (14)
\end{aligned}
$$

• Note that, this system is the same as the equations (4) - (7), except the **perturbation** $XS\mathbf{e} = \mu\mathbf{e}$ and $(x, s) > 0$.

• For a given μ, the system of nonlinear equations (11)-(14) provides a solution $(x_{\mu}, \lambda_{\mu}, s_{\mu})$.

• x_{μ} lies in interior of the feasible set of LOP, while the pair (λ_{μ}, s_{μ}) lies in the interior of the feasible set of LOP_{D}, due to $XS\mathbf{e} = \mu\mathbf{e}$ and $(x, s) > 0$. Furthermore,

Primal-dual Interior Method for LOPs...

- Furthermore, if

$$x^* = \lim_{\mu \searrow 0^+} x_\mu \text{ and } (\lambda^*, s^*) = \lim_{\mu \searrow 0^+} (\lambda_\mu, s_\mu)$$

the x^* is a minimum point of LOP, while (λ^*, s^*) is a maximum point of LOP_D.

- Therefore, any algorithm that solves the system of nonlinear equations (11)-(14) is known as a **primal-dual interior point algorithm**.
- For a given μ, to determine the triple $(x_\mu, \lambda_\mu, s_\mu)$,

(I) solve the nonlinear system $F_\mu(x, \lambda, s) = \begin{bmatrix} Ax - b \\ A^\top \lambda + s - c \\ XS\mathbf{e} - \sigma\mu\mathbf{e} \end{bmatrix} = 0,$

(II) and guarantee always that $(x, s) > 0$.

Primal-dual Interior Method for LOPs...

- The set of

$\mathcal{C} = \{(x(\mu), \lambda(\mu), s(\mu)) \mid F_\mu(x(\mu), \lambda(\mu), s(\mu)) = 0, (x(\mu), s(\mu)) > 0\}$

is known as the **central path**.

(I) To solve the system

$$F_\mu(x, \lambda, s) = \begin{bmatrix} Ax - b \\ A^\top \lambda + s - c \\ XS\mathbf{e} - \sigma\mu\mathbf{e} \end{bmatrix} = 0$$

use a Newton method.

- For a given μ and feasible point (x, λ, s), determine $d = (\Delta x, \Delta\lambda, \Delta s)$ by solving $J_\mu(x, \lambda, s)d = -F_\mu(x, \lambda, s)$; i.e.,

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ X & 0 & S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{bmatrix} = - \begin{bmatrix} Ax - b \\ A^\top \lambda + s - c \\ XS\mathbf{e} - \sigma\mu\mathbf{e} \end{bmatrix} \tag{15}$$

- Next iterate $(x^+, \lambda^+, s^+) = (x, \lambda, s) + \alpha(\Delta x, \Delta\lambda, \Delta s)$.

Primal-dual Interior Method for LOPs...

II: Question

How to guarantee that $(x_\mu, s_\mu) > 0$?

Answer

We know that $x_i s_i = \mu, i = 1, \dots, n.$ Hence,

$$x^\top s = x_1 s_1 + x_2 s_2 + \ldots + x_n s_n = n\mu \Rightarrow \frac{x^\top s}{n} = \mu$$

Therefore, choose μ so that $\frac{x^\top s}{n} > 0.$

Importance of the central path

- Additionally, for $(x_\mu, \lambda(\mu), s_\mu) \in \mathcal{C}$ we have $\frac{x^\top(\mu)s(\mu)}{n} = \mu.$
- Fast convergence of a PDIPM algorithm is achieved if iterates lie on the central path.
- The parameter σ is known as a centering parameter. Thus, σ is chosen to force iterates remain closed to (or on) the central path.

Primal-dual Interior Method for LOPs...

A primal-dual interior point algorithm (PDIPM):

Step 0: • Give an initial point (x_0, λ_0, s_0) with $(x_0, s_0) > 0$.

• Set $k \leftarrow 0$ and $\mu_0 = \frac{x_0^\top s_0}{n}$

Repeat:

• Choose $\sigma_k \in (0, 1]$;

• Solve the linear system (16) with $\mu = \mu_k$ and $\sigma = \sigma_k$ to obtain $(\Delta x_k, \Delta\lambda_k, \Delta s_k)$;

• Choose step-length $\alpha_k \in (0, 1]$

• and set

$$
\begin{aligned}
\bullet\ x_{k+1} &= x_k + \alpha_k \Delta x_k \\
\bullet\ \lambda_{k+1} &= \lambda_k + \alpha_k \Delta\lambda_k \\
\bullet\ s_{k+1} &= s_k + \alpha_k \Delta s_k.
\end{aligned}
$$

Until: Some termination criteria is satisfied.

Primal-dual Interior Method for LOPs...

Questions:

Q1: How to determine the step length α_k?

Q2: How to choose the centering parameter σ_k?

Q3: What is a suitable termination criteria?

Q4: How to solve the system of linear equations (16)?

Some strategies for step-length selection:

(a) Use $\alpha_k = 1, k = 1, 2, \ldots$. But, generally, not advised.

(b) Choose α_k so that

$$\begin{aligned} x_k + \alpha_k \Delta x_k &> 0 \\ s_k + \alpha_k \Delta s_k &> 0. \end{aligned}$$

Compute the largest α that satisfies these condition

$$\alpha_{max} = \min \left\{ \underbrace{\min \left\{ \frac{x_{k,i}}{-\Delta x_{k,i}} \mid \Delta x_{k,i} < 0 \right\}}_{\alpha_{x,max}}, \underbrace{\min \left\{ \frac{s_{k,i}}{-\Delta s_{k,i}} \mid \Delta s_{k,i} < 0 \right\}}_{=\alpha_{s,max}} \right\}$$

Then choose $\alpha_k = \min\{1, \eta_k \cdot \alpha_{max}\}$. Typically $\eta_k = 0.999$.

Primal-dual Interior Method for LOPs...

(c) Different step lengths for x and s may provide a better accuracy. So choose

$$\alpha_{k,x} = \min\{1, \eta_k \cdot \alpha_{\max,x}\} \text{ and } \alpha_{k,s} = \min\{1, \eta_k \cdot \alpha_{\max,s}\}$$

Use the following update $x_{k+1} = x_k + \alpha_{k,x}\Delta x_k$ and $(\lambda_{k+1}, s_{k+1}) = (\lambda_k, s_k) + \alpha_{k,s}(\Delta\lambda_k, \Delta s_k)$.

Some strategies for choice of centering parameter:

(a) $\sigma_k = 0, k = 1, 2, \ldots,$ - affine-scaling approach;

(b) $\sigma_k = 1, k = 1, 2, \ldots,$

(c) $\sigma_k \in [\sigma_{min}, \sigma_{max}] = 1, k = 1, 2, \ldots$ Commonly, $\sigma_{min} = 0.01$ and $\sigma_{max} = 0.75$ (path following method)

(d) $\sigma_k = 1 - \frac{1}{\sqrt{n}}, k = 1, 2, \ldots,$ (with $\alpha_k = 1$ - short-step path-following method)

Primal-dual Interior Method for LOPs...

Some termination criteria:

- Recall that, at a solution (x, s, λ) equation (12) should be satisfied

$$c = A^\top \lambda + s.$$

This is equivalent to

$$c^\top = \lambda^\top A + s^\top.$$

Multiplying both sides by x, we obtain $c^\top x = \lambda^\top \underbrace{Ax}_{=b} + s^\top x$.
$\Rightarrow c^\top x = b^\top x + s^\top x$. Hence, $s^\top x = c^\top x - b^\top x$.

- Hence,

$$s^\top x = c^\top x - b^\top x$$

$s^\top x$ is a **measure of gap** between the primal objective function $c^\top x$ and the dual objective function $b^\top \lambda$.

Primal-dual Interior Method for LOPs...

- The optimality condition LOP's demands that: optimal solutions should satisfy $c^\top x = b^\top x$.
- So the expression $\mu = \frac{s^\top x}{n} = \frac{c^\top x - b^\top x}{n}$ is known as a **measure of the duality gap** between LOP and LOP$_D$.

Termination

The algorithm can be terminated at iteration step k if the duality gap

$$\mu_k = \frac{x_k^\top s_k}{n}$$

is sufficiently small, say $\mu_k < \varepsilon$.

Primal-dual Interior Method for LOPs...

Solution strategies for the system of linear equations

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ X & 0 & S \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^\top \lambda - s \\ \mu \mathbf{e} - XS\mathbf{e} \end{bmatrix} \tag{16}$$

- The efficiency of the primal-dual interior point methods is highly dependent on the algorithm used to solve this $2n + m$ linear system.
- The choice of an algorithm depends on the structure and properties of the coefficient matrix $\begin{bmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ X & 0 & S \end{bmatrix}$.
- **Sometimes it may be helpful first to eliminate Δx and Δs and solve for $\Delta \lambda$ from the reduced system**

$$\left(AX^{-1}XA^\top\right)\Delta\lambda = AX^{-1}S\left(c - \mu X^{-1}\lambda\right) + b - Ax, \tag{17}$$

then to directly compute $\Delta s = c - A^\top \lambda - s - A^\top \Delta\lambda$ and $\Delta x = X^{-1}\left(\mu \mathbf{e} - XS\mathbf{e} - S\Delta s\right)$.