# **Lecture 7:** Büchi Automata

In this lecture we shall study finite automata as acceptors of infinite words. The study of such automata goes back to Büchi ([1]) and continues to be a topic of research ([5]).

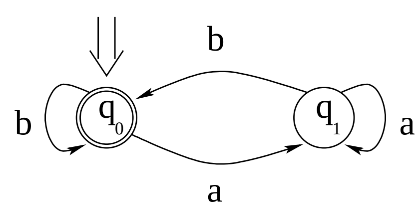
Let $\Sigma$ be a finite alphabet. An infinite word (or $\omega$-word) over $\Sigma$ is simply an infinite sequence $a_1a_2\ldots$ where each $a_i \in \Sigma$. We shall use $\Sigma^\omega$ to denote the set of all infinite words over the alphabet $\Sigma$.

Let $A = (Q, \Sigma, \delta, s, F)$ be a finite automaton. There is a natural generalization of the notion of a *run* from finite to infinite words. A *run* over an infinite word $\sigma = a_1a_2\ldots$ is a sequence $\rho = sq_1q_2\ldots$ with $s \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \ldots$. But, when is such a run *accepting*? One obvious choice is to define an accepting run as one that visits some state in $F$. But with such a definition one can't even describe the set of words which have infinitely many $a$s (Why?).

In any run $\rho$, some states of $Q$ are visited only finite number of times and some others are visited infinitely often. Let us call these sets *fin*$(\rho)$ and *inf*$(\rho)$. There is an (infinite) suffix of the run where none of the states from *fin*$(\rho)$ appear and the states from *inf*$(\rho)$ appear infinitely often. Thus, it is reasonable to assume that the classification of a run as accepting or rejecting must rely on its behaviour in the limit and hence must depend only on *inf*$(\rho)$. Büchi's suggestion was to classify a run as accepting if it visits the set $F$ infinitely often. Since there are only finitely many states in $Q$ and $F$, this is equivalent to demanding that the run visit some fixed state in $F$ infinitely often.

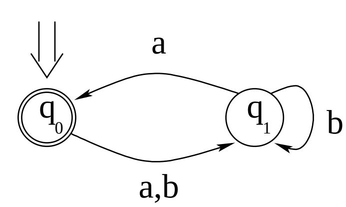
Formally, a *Büchi Automaton* is a finite automaton $A = (Q, \Sigma, \delta, s, F)$, and the language accepted by such an automaton is $L(A) = \{\sigma \mid \text{there is a run } \rho \text{ over } \sigma \text{ such that } inf(\rho) \cap F \neq \emptyset\}$. A langugage $L \subseteq \Sigma^\omega$ is said to be $\omega$-*regular* if it is accepted by some Büchi automaton.

The automaton



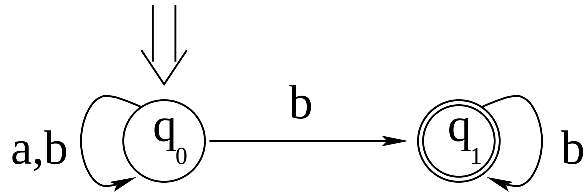

accepts all infinite words over $\{a, b\}$ in which, every $a$ has a $b$ occuring some where to its right. In this lecture we shall omit the $\omega$ and call a $\omega$-word as simply a word.

The following automaton




accepts all words that have infinitely many $a$s.

The automaton




accepts all words that have finitely many $a$s. Call this language $\mathsf{Finite(a)}$.

# 1 Deterministic Büchi Automata

Of the three automata described above the first two are deterministic whilst the last one is not. Can one design a deterministic Büchi automaton that accepts the set of words with finite number of $a$s? The answer is negative and this can be seen as follows: Suppose there is a deterministic Büchi automaton $A$ accepting this language. This automaton must have an accepting run on the word $ab^\omega$ (where $b^\omega = bbbb\ldots$). Suppose this (unique) run enters a state in F after $ab^{n_1}$ for some $n_1 \geq 1$. Now, $ab^{n_1}ab^\omega$ is also in the language and there is a unique run on this word, which extends the aforementioned run on $ab^{n_1}$, that is accepting and such a run must visit a state in $F$ after reading $ab^{n_1}ab^{n_2}$ for some $n_2 \geq 1$. Repeating this argument we can construct a sequence $ab^{n_1}ab^{n_2}\ldots ab^{n_i}\ldots$ on which the unique run visits a state in $F$ after reading $ab^{n_1}$, $ab^{n_1}ab^{n_2}$, $\ldots ab^{n_1}ab^{n_2}\ldots ab^{n_i}, \ldots$. Thus, this run visits the set $F$ infinitely often and hence this string with infinitely many $a$s is accepted by $A$. This contradicts our assumption that $A$ accepted the language of words with finite number of $a$s. Thus, nondeterministic Büchi automata are more powerful than deterministic Büchi automata.

Let $L$ be a regular language of finite words. We define $\widehat{L}$ to be the $\omega$-language consisting of all words that have infinitely many prefixes in $L$[1]. For example if $L = \Sigma^*.a$ then $\widehat{L}$ is the set of words with infinitely many $a$s. Then, we can characterize the class of languages accepted by deterministic Büchi automata as follows:

**Theorem 1** *Let $A$ be a deterministic Büchi automaton and let $L_f(A)$ be the language of finite words accepted by $A$ when treated as a finite automaton and let $L(A)$ be the language accepted by $A$ as a Büchi automaton. Then,*

$$L(A) \quad = \quad \widehat{L_f(A)}$$

The proof of this theorem quite easy and we leave it as an exercise. Our proof above showing that $\mathsf{Finite(a)}$ is not accepted by any deterministic Büchi automaton can be seen as showing that $\mathsf{Finite(a)}$ is not $\widehat{L}$ for any language $L$.

We now examine the closure properties of $\omega$-regular languages. Given Büchi automata recognizing languages $L_1$ and $L_2$ it is quite trivial to construct a Büchi automaton accepting the language $L_1 \cup L_2$. On the other hand constructing an automaton that accepts $L_1 \cap L_2$ requires some ingenuity. We leave that as an interesting exercise.

[1]If you wonder I write $\widehat{L}$ and not $\vec{L}$, the answer is rather embarassing. I can't get $\vec{L}$ to stretch over longer expressions like $L_1 \vec{\cup} L_2$

**Exercise:** Show how to construct a Büchi automaton accepting $L_1 \cap L_2$ from automata accepting $L_1$ and $L_2$.

**Exercise:** Show how to construct deterministic Büchi automata accepting $L_1 \cup L_2$ and $L_1 \cap L_2$ from deterministic automata accepting $L_1$ and $L_2$.

# 2 Complementation of Büchi Automata

We saw three different techniques to establish the closure under complementation of regular languages: via deterministic automata, via Myhill-Nerode congruences and finally via alternating automata. From the previous section it seems that the first route is not available in the case of $\omega$-regular languages. However, using more general acceptance conditions than the Büchi condition, one can obtain deterministic automata accepting all $\omega$-regular languages. In the next couple of lectures we shall use each of the three routes in demonstrating the closure under complementation of $\omega$-regular languages.

The easiest technique, and the one used by Büchi himself, is via congruences. Our presentation below follows that of Thomas [7]. Recall that we associated a congruence over $\Sigma^*$ with every deterministic finite automaton $A$, given by $x \equiv_A y$ if and only if $\forall q \in Q.\delta(q,x) = \delta(q,y)$. Since acceptance of finite words is decided by where the run starting at $s$ ends up, this is the right notion. This equivalence is of finite index and further it saturates $L(A)$. That is, $[x]_{\equiv_A} \subseteq L(A)$ or $[x]_{\equiv_A} \cap L(A) = \emptyset$ for each $x \in \Sigma^*$. Since each $x \in \Sigma^*$ lies in some class (namely $[x]$) we can write both $L(A)$ as well as $\overline{L(A)}$ as unions of these equivalence classes (each of which is a regular language).

How do we extend these ideas to $\omega$-regular languages? We need to extend the above construction in two ways, firstly we must be able associate a congruence with nondeterministic automata and secondly it must capture the notion of Büchi acceptance. The first step is easy: With any nondeterministic finite automaton $A$ we can associate a congruence $\equiv_A$ defined by $x \equiv_A y$ if and only if $\forall q.\ \delta(q,x) = \delta(q,y)$ where this equality is an equality of sets. In other words, there is a run from $q$ to $q'$ on the word $x$ if and only if there is a run from $q$ to $q'$ on the the word $y$ (for any $q$ and $q'$). This relation is a congruence, is of finite index and saturates $L(A)$. Extending this relation to capture accepting runs over infinite words requires a little bit of thought. If $x$ and $y$ are equivalent then we would like to be able to replace any number of occurances of $x$ with $y$ in any run without affecting acceptance. This leads us to the following definition: With each Büchi automaton $A$ we associate a relation $\equiv_A$ over $\Sigma^*$ given by

$$\begin{aligned} x \equiv_A y \quad \stackrel{\triangle}{=} \quad & \forall q,q'.\ q \xrightarrow{x} q' \iff q \xrightarrow{y} q' \quad \wedge \\ & \forall q,q'.\ q \xrightarrow{x}_g q' \iff q \xrightarrow{y}_g q' \end{aligned}$$

where $q \xrightarrow{x}_g q'$ means that there is a run from $q$ to $q'$ on the word $x$ that passes through some state in $F$ (there may be other runs that do not pass through any state in $F$). We shall often write $[x]$ for $[x]_{\equiv_A}$.

If $x$ and $y$ are equivalent then in any infinite run we may replace any number of subruns on $x$ by an appropriate runs on $y$ in such a way that if the original run was accepting then the modified run continues to be accepting.

It is easy to check that this relation is a congruence. It is of finite index as the number equivalence classes is bounded by the number of functions from $Q$ to $2^Q \times 2^Q$ (Why?). Further, we claim that for any $x$ and $y$, $[x].[y]^\omega \subseteq L(A)$ or $[x].[y]^\omega \cap L(A) = \emptyset$. This can be seen as follows: Suppose $x_1 y_1 y_2 \ldots \in L(A)$ with $x_1 \in [x]$ and $y_i \in [y]$ for $i$. Consider any accepting run $\rho = s \xrightarrow{x_1} q_1 \xrightarrow{y_1} q_2 \xrightarrow{y_2} q_3 \ldots$. Let $x'_1 \equiv x_1$, $y'_1 \equiv y_1$ and so on. Then, we know that there are runs $s \xrightarrow{x'_1} q_1$, $q_1 \xrightarrow{y'_1} q_2$ and so on, such that whenever the run $q_i \xrightarrow{y_i} q_{i+1}$ visits a final state so does the run $q'_i \xrightarrow{y_i} y'_{i+1}$. Thus, the run $s \xrightarrow{x'_1} q_1 \xrightarrow{y'_1} q_2 \xrightarrow{y'_2} q_3 \ldots$ visits $F$ infinitely often and hence $x'_1 y'_1 y'_2 \ldots$ is also in $L(A)$.

So what have we got so far? Suppose $\equiv$ has $N$ congruence classes. Then, each of the $N^2$ $\omega$-languages obtained as $[x].[y]^\omega$ is either completely contained in $L(A)$ or in $\overline{L(A)}$. Further, the following exercise guarantees that all these $N^2$ languages are $\omega$-regular.

**Exercise:** Let $U$ and $V$ be regular languages. Show that $U.V^\omega$ is a $\omega$-regular language.

So, it seems that we have shown that $\equiv$ "saturates" $L(A)$ and should be able to conclude that both $L(A)$ and $\overline{L(A)}$ are just finite unions of languages of the form $[x].[y]^\omega$. Then, using the above exercise we have a proof that the complement of a $\omega$-regular language is also $\omega$-regular. However, there is a gap. Unlike the case of finite words where it is a trival fact that each word in $\Sigma^*$ lies in some equivalence class of $\equiv$, it is not clear that every $\omega$-word is an element of $[x].[y]^\omega$ for some $x, y$. This needs proof and is in fact the most intricate part of Büchi's argument.

Following Gastin and Petit [4] (who attribute the original ideas to Perrin and Pin, see for instance, [8]), we shall find it convenient to use the monoid $\Sigma^*/\equiv_A$ and prove following general result about monoids.

**Theorem 2** *Let $M_1$ be the free monoid over a (possibly infinite) alphabet $\Sigma$. $M$ be a finite monoid and let $h$ be a homomorphism from $M_1$ to $M$. Let $a_1 a_2 a_3 \ldots$ be any infinite word over $\Sigma$. Then, there are elements $s$ and $e$ in $M$ such that, $e.e = e$ and $a_1 a_2 \ldots \in h^{-1}(s)(h^{-1}(e))^\omega$.*

**Proof:** For $i < j$, let $m(i,j)$ denote $h(a_i.a_{i+1}.\ldots.a_{j-1})$. This mapping is a colouring of all the 2-subsets of $N$ using the finite set $M$. Then, the infinite version of Ramsey's theorem guarantees that there is a subset $i_1, i_2, \ldots$ such that the colour of any 2-subset in this set is identical. That is $m(i_j, i_k) = m(i_l, i_m)$ for any $j < k, l < m$. Let $s = m(1, i_1)$ and $e = m(i_1, i_2)$. Then, $e.e = m(i_1, i_2).m(i_1, i_2) = m(i_1, i_2).m(i_2, i_3) = m(i_1, i_3) = e$. Finally, by the definition of $m(i,j)$, $a_1 \ldots a_{i_1 - 1} \in h^{-1}(s)$ and $a_{i_j} a_{i_j+1} \ldots a_{i_{j+1}} \in h^{-1}(e)$.

If you do not like the infinite version of Ramsey's theorem, here is a direct argument. Let $N_0 = \{1, 2, \ldots\}$. For $i = 0, 1, 2, \ldots$ we define the set $N_i$, the number $n_i$ and an element $s_i \in M$ as follows:

$$\begin{array}{lcl} n_i & = & \text{smallest number in } N_i \\ s_i & = & \text{some element of } M \text{ such that for infinitely many } j \in N_i,\ m(n_i, j) = s_i \\ N_{i+1} & = & \{j \mid m(n_i, j) = s_i\} \end{array}$$

Clearly, $N_i$ is infinite for each $i$. Let $e$ be an element that occurs as $s_i$ for infinitely many $i$ and let $i_1 < i_2 < \ldots$ be such that $s_{i_j} = e$. Then, $m(n_{i_j}, n_{i_k}) = e$ for any $j < k$. Thus, the indices $i_1, i_2, \ldots$ identify a Ramsey subset and the rest of the proof follows as above. Notice that since $n_0 = 1$, we also have that $s = m(1, n_{i_1}) = m(1, n_{i_2}) = \ldots$ and hence $s.e = s$. ∎

A pair of elements $(s, e)$ in a monoid with $s.e = s$ and $e.e = e$ is called a *linked pair*. Thus we have established that any infinite sequence over $M_1$ is in a set of the form $h^{-1}(s).(h^{-1}(e))^\omega$ for a linked pair $(s, e)$. It turns out that linked pairs play a rather important role in the study of the algebraic theory of $\omega$-regular languages, a topic to which we shall return later in the course. Notice that the second element of a linked pair is an idempotent and we shall exploit this fact shortly.

Using $\Sigma^*$ as $M_1$ and $\Sigma/{\equiv_A}$ as $M$ we can conclude that each $a_1a_2\ldots$ in $\Sigma^\omega$ is in $\eta^{-1}([x]).(\eta^{-1}([y]))^\omega$ where $\eta(x) = [x]$. But $\eta^{-1}([x]) = [x]$ and thus $a_1a_2\ldots \in [x].[y]^\omega$ for some $x, y$. Putting this together with our earlier calculations yields the following result:

**Theorem 3 (Büchi)** *Let $A$ be a Büchi automaton. Then,*

$$\begin{array}{rcl} L(A) & = & \bigcup_{\{x,y \ \mid \ [x][y]^\omega \subseteq L(A)\}} [x][y]^\omega \\ \overline{L(A)} & = & \bigcup_{\{x,y \ \mid \ [x][y]\omega \not\subseteq L(A)\}} [x][y]^\omega \end{array}$$

We also obtain the following characterization of $\omega$-regular languages:

**Theorem 4** *([8]) A language $L$ over $\Sigma^\omega$ is a $\omega$-regular language if and only if there is a homomorphism $h$ from $\Sigma^*$ to a finite monoid $M$ and a colletion $X$ of linked pairs over $M$ such that $L = \bigcup_{(s,e)\in X} h^{-1}(s).(h^{-1}(e))^\omega$.*

One direction of this theorem is proved above and the other direction is left as a (rather trivial) exercise.

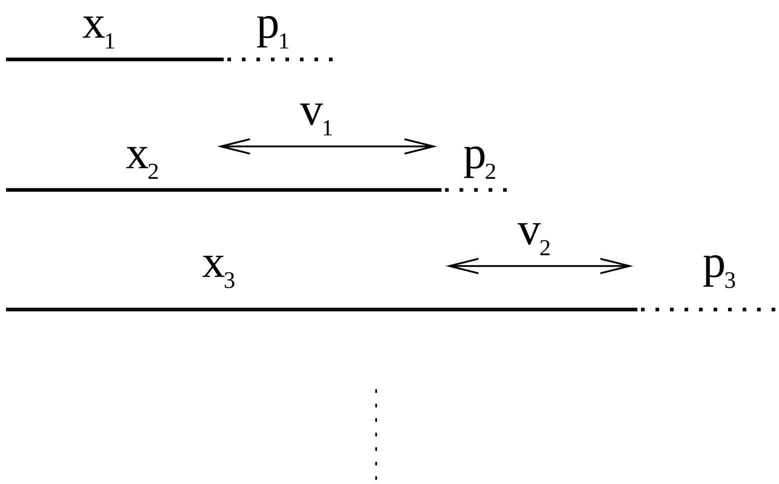
# 3 Determinizing Büchi Automata

We shall follow the route taken by Eilenberg and Schutzenberger [**?**], Choueka [2] as well as Rabin [9] as described by Perrin and Pin in [8]. Recall that deterministic Büchi automata recognize limit languages (i.e. languages of the form $\widehat{L}$ for some regular language $L$). We next show that if $h$ is any homomorphism from $\Sigma^*$ to a finite monoid $M$ and $e$ is an idempotent in $M$ then $(h^{-1}(e))^\omega$ is a limit language.

Let $X_e$ denote $h^{-1}(e)$ and let $P_e$ denote the prefix minimal words in $X_e$ (i.e. $P_e = \{x \in X_e \mid y < x \text{ implies } y \notin X_e\}$. Let $\sigma \in X_e^\omega$. Then, $\sigma = x_1x_2\ldots$ with each $x_i \in X_e$. For each $i$ there is a $p_i \in P_e$ with $p_i \leq x_i$. Thus, there are infinite many prefixes of the word $\sigma$, namely $x_1p_2,\ x_1x_2p_3,\ \ldots,\ x_1x_2\ldots x_ip_{i+1}, \ldots$, in $X_e.P_e$. Thus $X_e^\omega \subseteq \widehat{X_eP_e}$. The following lemma establishes the converse.

**Lemma 5** *Let $h$ be a homomorphism from $\Sigma^*$ to a finite monoid $M$ and let $e$ be an idempotent and let $X_e$ and $P_e$ be as defined above. Then, $\widehat{X_e.P_e} = X_e^\omega$.*

**Proof:** The discussion above established that $X_e^\omega \subseteq \widehat{X_e.P_e}$. Let $\sigma \in \widehat{X_e.P_e}$. Then, there are infinitely many $x_i$s in $X_e$ and $p_i$s in $P_e$ such that $x_1p_1 < x_2p_2 < \ldots x_ip_i \ldots < \sigma$. Suppose the lengths of these $x_i$s were bounded. Then for some $i < j$, $x_i = x_j$ and this in turn means that $x_ip_i < x_ip_j$ contradicting the requirement that $p_j$ has no prefixes in $X_e$. Thus, without loss of generality we may assume that $x_ip_i < x_{i+1}$. Let us define $v_1, v_2, \ldots$ as words such that $x_iv_i = x_{i+1}$. Note that $p_i < v_i$.




Thus, $\sigma = x_1v_1v_2v_3\ldots$. Now, we use the fact that $x_1, v_1, v_2, \ldots$ are elements of the monoid $\Sigma^*$ and Theorem 2 to conclude the existence of $s$ and $f$ in $M$ such that $x_1v_1v_2\ldots$ can be factored as $h^{-1}(s).h^{-1}(f)^\omega$ with $f.f = f$ and $s.f = s$. Since we treat $x_1, v_1, v_2, \ldots$ as elements of the monoid $\Sigma^*$ the factoring respects these word boundries. That is, there is a factorization where $h^{-1}(s)$ looks like $x_1v_1\ldots v_i$ and $v_{i+1}\ldots v_j$ is in $h^{-1}(f)$ for some $j > i$ and so on. Then, $h(x_1v_1\ldots v_i) = h(x_{i+1}) = e$. Thus $s = e$.

From the factoring, we have that $v_{i+1}\ldots v_j \in h^{-1}(f)$ and since $p_{i+1} < v_{i+1}$ there must be a $g = h(v_{i+1}\ldots v_j/p_{i+1})$ (where $x/y$ is $z$ if $x = yz$) with $eg = f$.

Here is a cute fact about idempotents. If $e$ and $f$ are idempotents and $eg = f$ then $ef = f$. In proof note that $ef = eeg = eg = f$. Thus, $ef = f$. But recall that $s.f = s$ and $s = e$ and thus $e.f = e$. Thus $e = f$ we have actually factored $\sigma = x_1v_1v_2\ldots$ as $(h^{-1}(e))^\omega$ and thus $\sigma \in X_e^\omega$. ■

Now, putting together Theorem 4 and Lemma 5 we have the following characterization of $\omega$-regular languages.

**Theorem 6** *A language $L$ is $\omega$-regular if and only if there is a finite set $I$ and regular languages $U_i, V_i$, for each $i \in I$, such that*

$$L = \bigcup_{i\in I} U_i.\widehat{V_i}$$

Actually, there is a monoid $M$ and a homomorphism $h$ to $M$ and a a linked pair $(s_i, e_i)$ for each $i$ such that $U_i$ is recognized via $s_i$ and $V_i$ via $e_i$. Note, that there is no hope of replacing $U_i\widehat{V_i}$ by some $\widehat{W_i}$ (Why? Use the fact that limit languages are closed under union).

**Notes:** The most widely used introductary article on $\omega$-automata is [7]. Another good introduction to this topic is found in [6]. Our presentation has drawn from [7] and [8].

# References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