Fourier Transforms and Frequency-Domain Processing

Supplemental Slides

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Outline

Ch. 5.2: Authors' five key concepts

1. The harmonic content of signals.
   - For this class, "signal" means "image."
2. The Fourier representation is a complete alternative.
   - There are two well-known caveats that the text does not mention. One is not important in practice, the other one is common.
3. Fourier processing concerns the relation between the harmonic content of the output signal and the harmonic content of the input signal.
4. The space domain and Fourier domain are reciprocal.
5. The Fourier series expansion and the Fourier transform have the same basic goal.
   - One is for periodic signals, the other is for aperiodic.
   - One can be thought of as the limiting case of the other.

Ch. 5.2: Authors' five key concepts

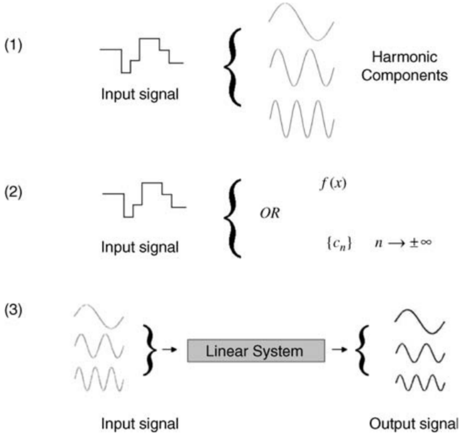


Figure 1: Illustration of the authors' first 3 key concepts. This is Figure 5.2 on p.115 of the text.

Ch. 5.2: Authors' five key concepts

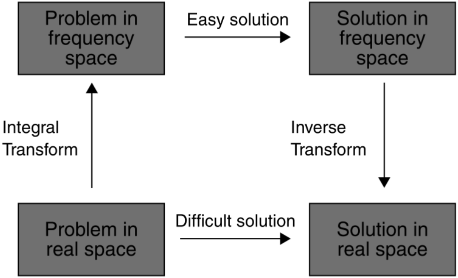


Figure 2: Illustration of key concept #3. This is Figure 5.1 in the text, p.114.

Ch. 5.2.1: The Fourier series

Key Point 1

Any *periodic* signal may be expressed as a weighted combination of sine and cosine functions having different periods or frequencies.

Definition (Real Fourier series expansion of time-domain signal)

A periodic signal V(t) having period T can be constructed exactly as an infinite sum of harmonic functions, a Fourier series, as follows:

$$V(t) = \sum_{i=0}^{\infty} a_n cos\left(\frac{2\pi nt}{T}\right) + \sum_{i=1}^{\infty} b_n sin\left(\frac{2\pi nt}{T}\right)$$
$$= \sum_{i=0}^{\infty} a_n cos(\omega_n t) + \sum_{i=1}^{\infty} b_n sin(\omega_n t)$$

Ch. 5.2.1: The Fourier series

Definition (Real Fourier series expansion of space-domain signal)

A periodic 1D function of a function of a *spatial* coordinate $f(x)$ having spatial period λ can be represented the same way[a].

$$f(x) = \sum_{i=0}^{\infty} a_n cos\left(\frac{2\pi nx}{\lambda}\right) + \sum_{i=1}^{\infty} b_n sin\left(\frac{2\pi nx}{\lambda}\right)$$

$$= \sum_{i=0}^{\infty} a_n cos(k_n x) + \sum_{i=1}^{\infty} b_n sin(k_n x)$$

[a] The authors use $V(x)$ in this formula instead of $f(x)$. I believe this is a mistake.

Ch. 5.2.1: The Fourier series

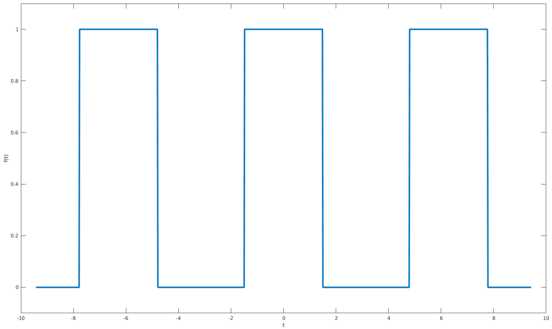
Simple observations, p.116 of the text:

1. The infinite series of harmonic sums in this expansion, namely $cos(k_n x)$ and $sin(k_n x)$ are called the Fourier *basis functions*.
2. We are dealing with a function that varies in space and the (inverse) periodicity, determined by $k_n = 2\pi n/\lambda$, is called the *spatial frequency*[1].
3. The coefficients a_n and b_n indicate how much of each basis function is required to "build" $f(x)$. The complete set of coefficients constitute the *Fourier* or *frequency spectrum* of the spatial function.
4. To reproduce the original function $f(x)$ *exactly*[2], the expansion must extend to an infinite number of terms.

[1] This is often called the *angular frequency* because the units are radians per unit length, and "spatial frequency" refers instead to $1/\lambda$.

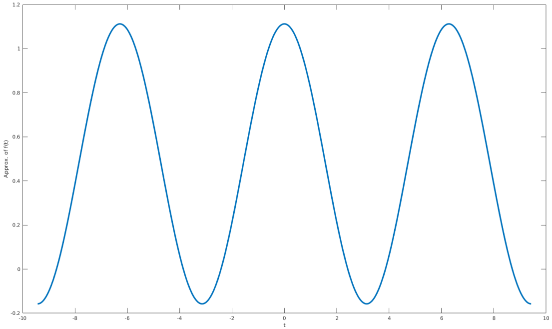
[2] There is a slight caveat here, that matters in practice, but we can ignore it for now.

Square wave approximation by sum of cosines



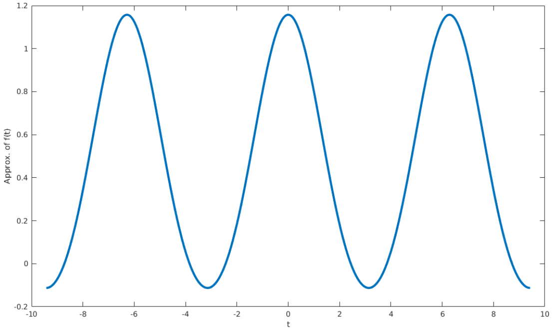
Square wave, denoted by $f(t)$, period $T_0 = 2\pi$

Square wave approximation by sum of cosines



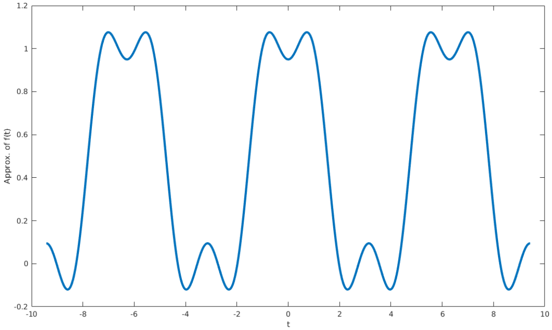
$$\hat{f}(t) = .48 + .62 \cos \frac{2\pi}{T_0} t$$

Square wave approximation by sum of cosines



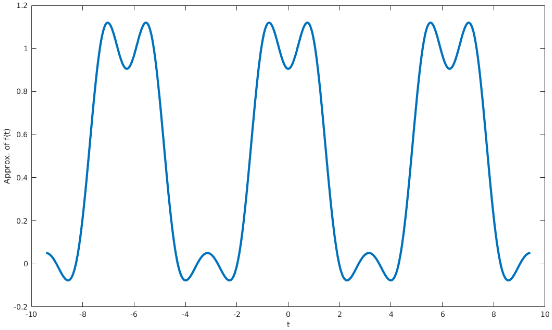
$$\hat{f}(t) = .48 + .62\cos\frac{2\pi}{T_0}t + .045\cos\frac{4\pi}{T_0}t$$

Square wave approximation by sum of cosines



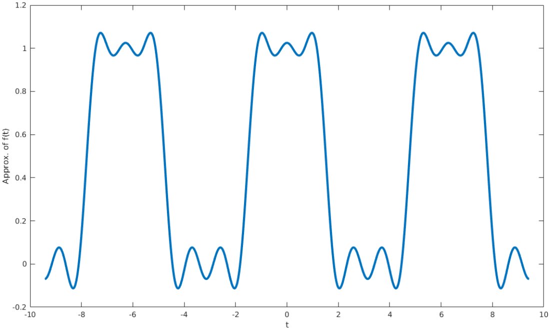
$$\hat{f}(t) = .48 + .62 \cos \frac{2\pi}{T_0} t + .045 \cos \frac{4\pi}{T_0} t - .21 \cos \frac{6\pi}{T_0} t$$

Square wave approximation by sum of cosines



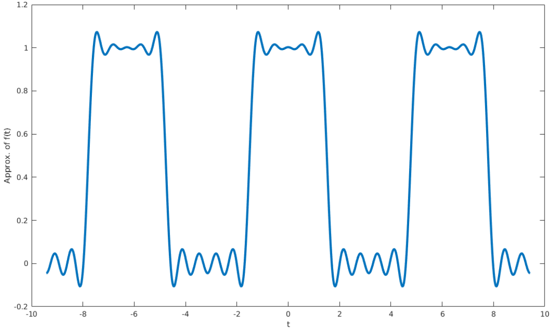
$$\hat{f}(t) = .48 + .62\cos\frac{2\pi}{T_0}t + .045\cos\frac{4\pi}{T_0}t - .21\cos\frac{6\pi}{T_0}t - .45\cos\frac{8\pi}{T_0}t$$

Square wave approximation by sum of cosines



$$\hat{f}(t) = .48 + .62 \cos \frac{2\pi}{T_0} t + .045 \cos \frac{4\pi}{T_0} t - .21 \cos \frac{6\pi}{T_0} t - .45 \cos \frac{8\pi}{T_0} t + .12 \cos \frac{10\pi}{T_0} t$$

Square wave approximation by sum of cosines



$$\hat{f}(t) = .48 + .62\cos\frac{2\pi}{T_0}t + .045\cos\frac{4\pi}{T_0}t - .21\cos\frac{6\pi}{T_0}t - .45\cos\frac{8\pi}{T_0}t$$

$$+ .12\cos\frac{10\pi}{T_0}t + .044\cos\frac{12\pi}{T_0}t - .08\cos\frac{14\pi}{T_0}t - .043\cos\frac{16\pi}{T_0}t + .057\cos\frac{18\pi}{T_0}t$$

Ch. 5.2.1: The Fourier series

Key Point 2

The Fourier spectrum is a valid and complete alternative representation of a function.

Observations from our experiment with the square wave[3]:

- *Low frequencies* (corresponding to lower values of n) build the basic smooth shape.
- High frequencies needed for sharp transitions / edges.
- A good approximation to a periodic function can often be obtained with a *finite and relatively small number of frequencies*. We can define a *frequency cut-off* $k_{CO} = 2\pi N/\lambda$

[3] The text, p. 117, makes these observations about a step function. They are the same for a square wave. I thought that a square wave was a better example because it is easier to see the periodicity of a square wave.

Ch. 5.3: Calculation of the Fourier spectrum

By exploiting the *orthogonality* properties of the Fourier basis[4], we can obtain simple formulae for coefficients:

Definition (Real Fourier series coefficients)

$$a_n = \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(x) \cos(k_n x) dx$$

$$b_n = \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(x) \sin(k_n x) dx$$

[4] The authors explain this with a long proof, but if we use the *complex* Fourier series, it becomes much easier.

Ch. 5.4: Complex Fourier series

We may exploit the orthogonality relations between complex exponential functions to obtain the Fourier expansion coefficients:[5]

Definition (Complex Fourier series coefficients)

$$c_n = \frac{1}{\lambda}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(x)\exp(ik_n x)dx$$

[5] The authors refer us to the text's web page, but the content is "to follow".

Ch. 5.4: Complex Fourier series

It is relatively straightforward to show that the complex coefficients c_n are related to the real coefficients a_n and b_n in the real Fourier series:

Theorem (Relation between real and complex Fourier series)

$$\begin{aligned} c_k &= a_k + ib_k \\ c_{-k} &= a_k - ib_k \quad \text{for } k = 0, 1, \ldots, \infty \end{aligned}$$

Ch. 5.5: The 1-D Fourier transform

Definition (1D Fourier transform)

$$F(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-i\mu x) dx$$

Differences in convention:

- Here, I divide by $\sqrt{2\pi}$, but the authors divide by 2π instead. This is mathematically correct because the authors' definition of the inverse Fourier transform is consistent, but this is annoyingly *inconsistent* with the text's definition of the 2D Fourier transform.
- The authors sometimes refer to the frequency variable as k, and other times as k_x. They sometimes use k_n to refer to an angular frequency. To avoid confusion, I use μ.

Ch. 5.5: The 1-D Fourier transform

The Fourier transform $F(\mu)$ is a *complex function* and we can, therefore, also write the Fourier spectrum in polar form as the product of the Fourier modulus and (the exponential of) the Fourier phase:

Theorem (Fourier spectrum in polar form)

$$F(\mu) = |F(\mu)| \exp(-i\varphi(\mu))$$

Definition (Fourier modulus)

$$|F(\mu)| = (\text{Re}\{z\})^2 + (\text{Im}\{z\})^2$$

Definition (Fourier phase)

$$\varphi(\mu) = \arctan \frac{\text{Im}\{\mu\}}{\text{Re}\{\mu\}}$$

Ch. 5.6: The inverse Fourier transform and reciprocity

Definition (1D inverse Fourier transform)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu) \exp(i\mu x) d\mu$$

Differences in convention:

- Here, I divide by $\sqrt{2\pi}$, but the authors do not. This is mathematically correct because the author's definition of the Fourier transform is consistent, but this is annoyingly *inconsistent* with the text's definition of the 2D Fourier transform.

Ch. 5.6: The inverse Fourier transform and reciprocity

Key Point 4

The space domain and the Fourier domain are reciprocal.

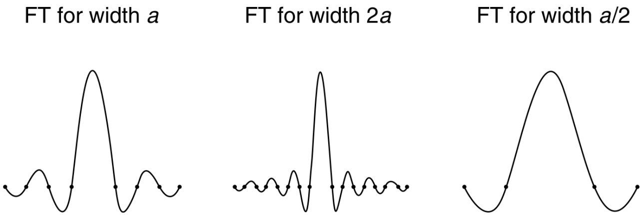


Figure 3: The Fourier transform of the rectangle function is the sinc function: $F(\mu) = (a/2\pi)(\sin(\mu a/2)/(\mu a/2)$. This is Figure 5.4, p.122 of the text.

Ch. 5.7: The 2-D Fourier Transform

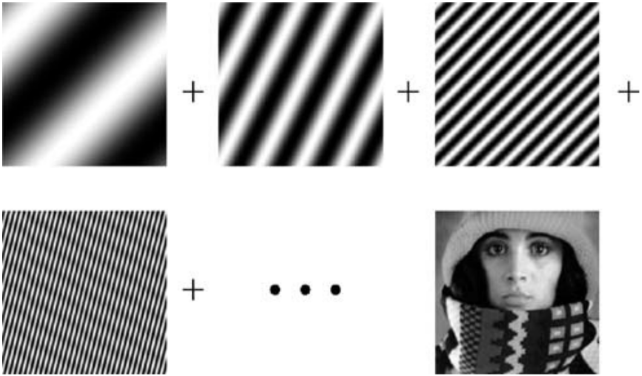


Figure 4: The central meaning of the 2D Fourier transform is that some scaled and shifted combination of the 2D harmonic basis functions can synthesize an arbitrary spatial function. This is Figure 5.5, p.125 of the text.

Ch. 5.7: The 2-D Fourier Transform

Definition (2D Fourier transform)

$$F(\mu,\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,y) \exp(-i(\mu x + \nu y)) dxdy$$

Definition (2D inverse Fourier transform)

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu,\nu) \exp(i(\mu x + \nu y)) d\mu d\nu$$

Differences in convention:

- I use μ and ν, while the authors use k_x and k_y, respectively, for reasons discussed on slide 14.