Lecture II.

Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi}{T}nt} \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad n = 0, \pm 1, \pm 2, \ldots$$

for a function $f(t)$ of a continuous variable t, that is periodic with period T.

Use eulers formula, $e^{j\theta} = \cos\theta + j\sin\theta$ to see that this expands to sines and cosines.

Functions that are not periodic (but whose area under the curve is finite) can be expressed as an integral of sines/cosines multiplied by a weighting function - we get the fourier transform.

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\,dt$$

where μ is now also continuous, t is integrated out, so this is simply a function of μ

$$F(\mu) = \mathcal{F}\{f(t)\}$$

The inverse transform gives back our function,

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,d\mu$$

From Euler's

$$F(u) = \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi u t) - j\sin(2\pi u t)\right] dt$$

if $f(t)$ is real, $F(u)$ is in general complex.

- units of u depend on units of t

t = seconds $\Leftrightarrow$ u = Hz

t = meters $\Leftrightarrow$ u = cycles/meter

t = pixels $\Leftrightarrow$ u = cycles/pixel

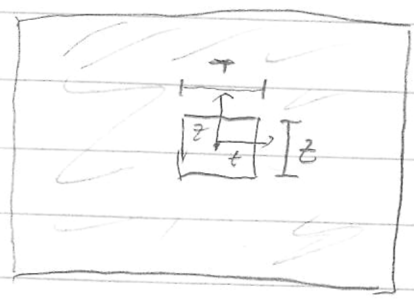
- 2-D Continuous Fourier Transform

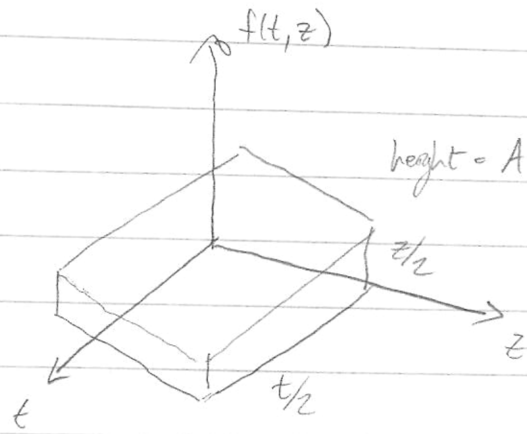
$f(t,z)$ - a continuous function of two continuous variables t, and z.

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)\, e^{-j2\pi(ut+vz)}\, dt\, dz$$

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ut+vz)}\, du\, dv$$

example - Fourier transform of a 2D box image





$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(ut+vz)}\,dt\,dz$$

$$= \int_{-T/2}^{T/2}\int_{-Z/2}^{Z/2} A e^{-j2\pi(ut+vz)}\,dt\,dz$$

$$= \int_{-T/2}^{T/2} Ae^{-j2\pi ut}\left[\int_{-Z/2}^{Z/2} e^{-j2\pi vz}\,dz\right]dt$$

$$\Downarrow$$

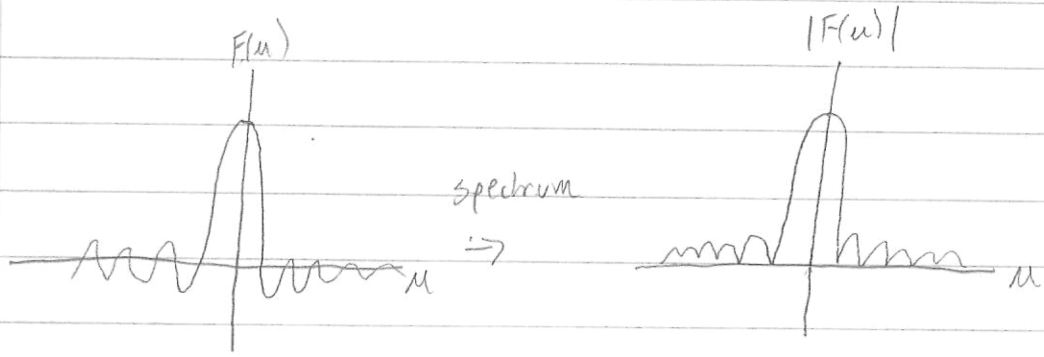
$$= \quad \frac{-1}{j2\pi v}\left[e^{-j2\pi vz}\right]_{-Z/2}^{Z/2} = \frac{1}{j2\pi v}\left[e^{j\pi vZ} - e^{-j\pi vZ}\right]$$

$$= Z\,\frac{\sin(\pi vZ)}{(\pi vZ)}$$

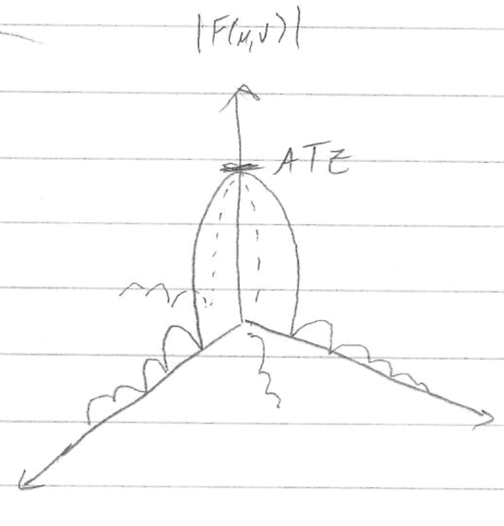
$$= Az \frac{\sin(\pi v z)}{\pi v z} \int_{-T/2}^{T/2} e^{-j2\pi u t}\, dt$$

$$= ATz \left(\frac{\sin(\pi v z)}{(\pi v z)}\right)\left(\frac{\sin(\pi u T)}{(\pi u T)}\right)$$

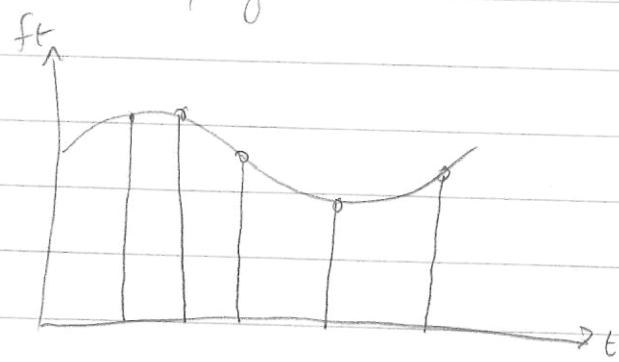
sinc   sinc



3D spectrum



what does sampling Do?



$$\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$$

Fourier transform of product of two functions in spatial domain is the convolution of the transforms in the freq. domain.

$$\tilde{F}(\mu) = \mathcal{F}\{\tilde{f}(t)\} = \mathcal{F}\{f(t)\, s_{\Delta T}(t)\}$$
$$= (F * S)(\mu)$$

see ex. 4.2 in Digital Im. Proc. Text book

$$S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

perform convolution integral in freq domain: (pg. 263)

$$\tilde{F}(\mu) = (F * S)(\mu) = \int_{-\infty}^{\infty} F(\tau)\, S(\mu - \tau)\, d\tau$$
$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

"The Fourier transform $\tilde{F}(\mu)$ of the sampled function $\tilde{f}(t)$ is an infinite, periodic sequence of copies of the transform of the original, continuous function"

The separation between copies is $\frac{1}{\Delta T}$

Separation is guaranteed if $\frac{1}{\Delta T} > 2\mu_{max}$

Nyquist rate

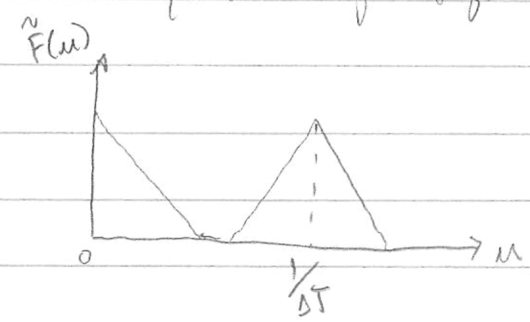
$\mu = \frac{1}{2\Delta T}$

To get the Fourier transform directly from the sampled function, we need the discrete Fourier transform.
Take M equally spaced samples over one period

$$F(\mu) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi\mu x/M} \qquad \mu = 0, 1, 2, \ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{\mu=0}^{M-1} F(\mu) e^{j2\pi\mu x/M} \qquad x = 0, 1, 2, \ldots, M-1$$

Due to these sampling procedures, the transform data in the interval from $[0, M-1]$ consists of two half periods meeting at $M/2$ where lower part is higher freq.



so we need to shift

Shifting in frequency domain:

$$f(x)e^{j2\pi\left(\frac{u_0 x}{M}\right)} \Leftrightarrow F(u-u_0)$$

let $u_0 = M/2 \rightarrow e^{j\pi x} = (-1)^x$ because $x \in \mathbb{Z}$

so $f(x)(-1)^x \Leftrightarrow F(u - M/2)$

in 2D: $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$