On Two Dimensional Fourier Analysis

- Image Processing is all about representation.
- So far in this class, representation in the spatial domain has been front and center because that is generally how we experience images.
- In this domain, convolution is the main tool for linear filtering. Later you'll explore nonlinear techniques (morphological operators).
- Another useful representation, albeit unnatural, is the image in the frequency domain. The main concept/result that makes this transformation
- useful is that convolution in the spatial domain is equivalent to multiplication in the frequency domain and vice-versa,

$$f(t) * h(t) \iff H(u)F(u)$$

$$f(t)h(t) \iff H(u) * F(u)$$

For large images this is a useful property.

- To get there we need a few concepts.

Fourier Series

Fourier, working on the mathematics describing the movement of heat with equations of the form

$$K_1 \frac{d^2x}{dt} + K_2 \frac{dx}{dt} = \emptyset$$

relied on periodic solutions

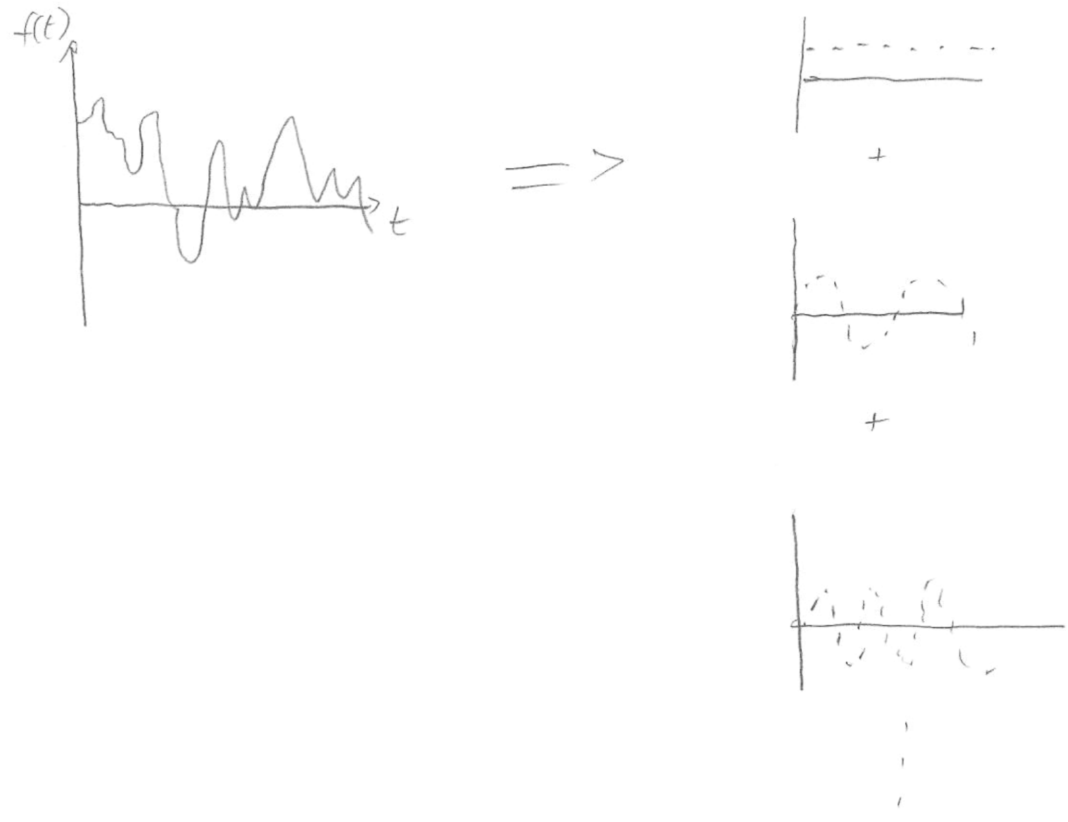
$$x(t) = A\cos w_0 t + B\sin w_0 t$$

- made the discovery that any periodic function can be expressed as a series of sines & cosines!

$$f(t) = a_0 + a_1 \cos \omega t + b_1 \sin \omega t$$
$$+ a_2 \cos 2\omega t + b_2 \sin 2\omega t$$
$$+ \;\; \ldots \;\; + \;\; \ldots$$

$$\omega = \frac{2\pi}{T}$$

- we would like to go the other way - given some arbitrary periodic function, what coefficients (a_i, b_i) do we need to describe the function in terms of sines & cosines? This would result in a different representation of the function $f(t)$



a_0 is easy since over a period, T, all the functions with sine or cosine average zero

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \quad \rightarrow \quad a_0 \text{ is just the average value}$$

for the rest we use fourier's "trick" (from feynman lectures)

ex. multiply both sides by $\cos 7\omega t$

$$\begin{aligned} f(t)\cdot\cos 7\omega t = {} & a_0 \cdot \cos 7\omega t \\ & + a_1 \cdot \cos\omega t \cdot \cos 7\omega t + b_1 \sin\omega t \cdot \cos 7\omega t \\ & + \quad \ldots \qquad\qquad + \quad \ldots \\ & + a_7 \cos 7\omega t \cdot \cos 7\omega t + b_7 \sin 7\omega t \cdot \cos 7\omega t \\ & \quad \ldots \qquad\qquad\qquad \ldots \end{aligned}$$

use identity $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

so the a_1 term becomes: $\frac{1}{2}a_1(\cos 8\omega t + \cos 6\omega t)$

perform the same averaging over T and you get exactly zero

but the a_7 term:

$$\frac{1}{2}a_7(\cos 14\omega t + \cos\emptyset) = \frac{1}{2}a_7$$

so $\text{Average}[f(t)\cdot\cos 7\omega t\, dt] = \frac{a_7}{2}$. like a sieve!

$$a_7 = \frac{2}{T}\int_0^T f(t)\cdot\cos 7\omega t\, dt$$

$$b_7 = \frac{2}{T}\int_0^T f(t)\cdot\sin 7\omega t\, dt$$

So in summary

1. $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$

2. $\int_0^T \sin n\omega t \cos m\omega t \, dt = \varnothing$ for all m, n

3. $\int_0^T \cos n\omega t \cos m\omega t \, dt = \int_0^T \sin n\omega t \sin m\omega t \, dt = \begin{cases} \varnothing & \text{if } n \neq m \\ \frac{T}{2} & \text{if } n = m \end{cases}$

4. $a_0 = \frac{1}{T} \int_0^T f(t) \, dt$

$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t \, dt$

$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t \, dt$

Fourier coefficients

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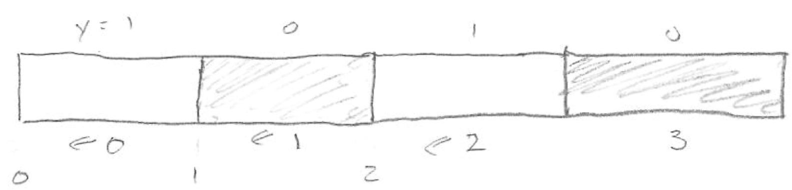
example:

take the following 1D image



T = 2

For integrals we can deal with piecewise continuous functions.

$a_0 = \frac{1}{2} \left[\int_0^1 1 \, dt + \int_1^2 0 \, dt \right] = \frac{1}{2}$

$a_n = \frac{2}{2} \left[\int_0^1 \cos n\pi t \, dt + \int_1^2 \cos n\pi t \cdot 0 \, dt \right] = \left[\frac{1}{n\pi} \sin n\pi t \right]_0^1 + \cancel{\left[\frac{1}{n\pi} \sin n\pi t \right]_1^2} = \varnothing$

$b_n = \frac{2}{2} \left[\int_0^1 \sin n\pi t \, dt + \cancel{\int_1^2 \sin n\pi t \, dt} \right] = \left(\frac{-1}{n\pi} \cos n\pi t \right)_0^1 + \cancel{\left(\frac{1}{n\pi} \cos n\pi t \right)_1^2}$

$-\frac{1}{n\pi}(\cos \pi n - 1) = \frac{1 - \cos \pi n}{n\pi}$ $\qquad \cos 2\pi n - \cos \pi n$ $\qquad \frac{\cos \pi n - 1}{n\pi}$

$$b_n = -\frac{1}{n\pi}\cos n\pi t \Big|_0^1 = -\frac{1}{n\pi}(\cos \pi n - 1) = \frac{1-\cos \pi n}{n\pi}$$

$$= \frac{1-(-1)^n}{n\pi}$$

$$b_n = \begin{cases} \frac{2}{n\pi} & \text{for } n \text{ odd} \\ \emptyset & \text{for } n \text{ even} \end{cases}$$

Sinc

$$f(t) \sim \frac{2}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} = \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right)$$

"Gibb's phenomenon"

∘ more generally a function can be considered periodic with period $\to \infty$ and given eulers identity $e^{jw} = \cos w + j\sin w$ we get the Fourier transform

$$F(u) = F(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ut}\,dt$$

$$f(t) = F^{-1}(F(u)) = \int_{u=-\infty}^{\infty} F(u)e^{j2\pi ut}\,du$$

t - in seconds, u in Hertz $\left(\frac{1}{\text{seconds}}\right)$

Convolution

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

$$F\{f(t) * h(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \right] e^{-j2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ut} dt \right] d\tau$$

let $t' = t - T$ $\quad F\{h(t-\tau)\} = H(u) e^{-j2\pi u\tau}$

$$= \int_{-\infty}^{\infty} f(\tau) H(u) e^{-j2\pi u\tau} d\tau$$

$$= H(u) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi u\tau} d\tau = H(u) F(u)$$