Week 6

Fourier Transform

consider a very simple function:

$$f(x) = \sin(2\pi x)$$

See CS6640_Week6

sample it at 8 points across $[0, 2\pi]$

$$\bar{f} : [0, -0.6, -0.96, -0.94, -0.54, 0.08, 0.66, 0.98]^T$$

The discrete 1D FT computes freq. domain ~~coefficients for~~ f:

$$F(u) = \frac{1}{8} \sum_{x=0}^{8-1} f(x) e^{-j2\pi u x / 8}$$

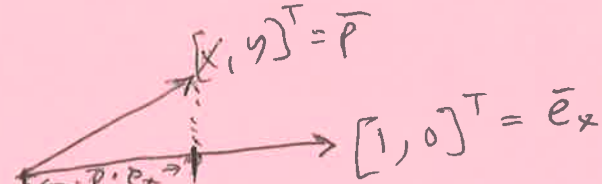
What is this?

let $\bar{B}_u = [e^{-j2\pi u \cdot 0/8}, e^{-j2\pi u \cdot 1/8}, \ldots, e^{-j2\pi u \cdot 7/8}]^T$

Then $F(u) = \frac{1}{m} \bar{f} \cdot \bar{B}_u$

i.e., the coefficient is how much $\bar{f}$ projects onto $\bar{B}_u$

e.g.



$$[x, y]^T \cdot [1, 0]^T = x + 0 = x$$

So, $\bar{B}_0 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

$f' \cdot \bar{B}_0 = -1.3216 + 0i$

$= \text{sum}(f)$

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look at the 8 basis vectors: they are complex

(see plots in CS6640_Week6)

Another way to look at it:

complex number: $z = a + bi$

let $r = \sqrt{a^2 + b^2}$

$\varphi = \text{atan2}(b, a)$

then $z = r(\cos\varphi + i\sin\varphi)$

$= re^{i\varphi}$

So, the dot product can now be viewed (over complex numbers) as vector addition

CS6640_Week6 $\bar{B}_k$ vectors

CS6640 week 6

See chapter 6 of Gonzalez & Woods 4th edition.

<u>inner product</u>

Euclidean: $\langle \bar{u}, \bar{v} \rangle = \bar{u}^T \bar{v} = \sum_{i=0}^{N-1} u_i v_i$

Unitary (complex) $\langle \bar{u}, \bar{v} \rangle = \bar{u}^{*T} \bar{v} = \sum_{i=1}^{N-1} u_i^* v_i = \langle \bar{v}, \bar{u} \rangle^*$

$\|\bar{z}\| = \sqrt{\langle \bar{z}, \bar{z} \rangle}$

$\theta = \cos^{-1} \frac{\langle \bar{z}, \bar{w} \rangle}{\|\bar{z}\| \|\bar{w}\|}$

$\bar{z}, \bar{w}$ orthogonal iff $\langle \bar{z}, \bar{w} \rangle = 0$

set $w_0, w_1, w_2, \ldots$ mutually orthogonal iff

$$\langle \bar{w}_k, \bar{w}_l \rangle = 0 \quad \forall k \neq l$$

form orthogonal basis of space they span

orthonormal if

$$\langle \bar{w}_k, \bar{w}_l \rangle = \delta_{kl} = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$$

Consider FT:

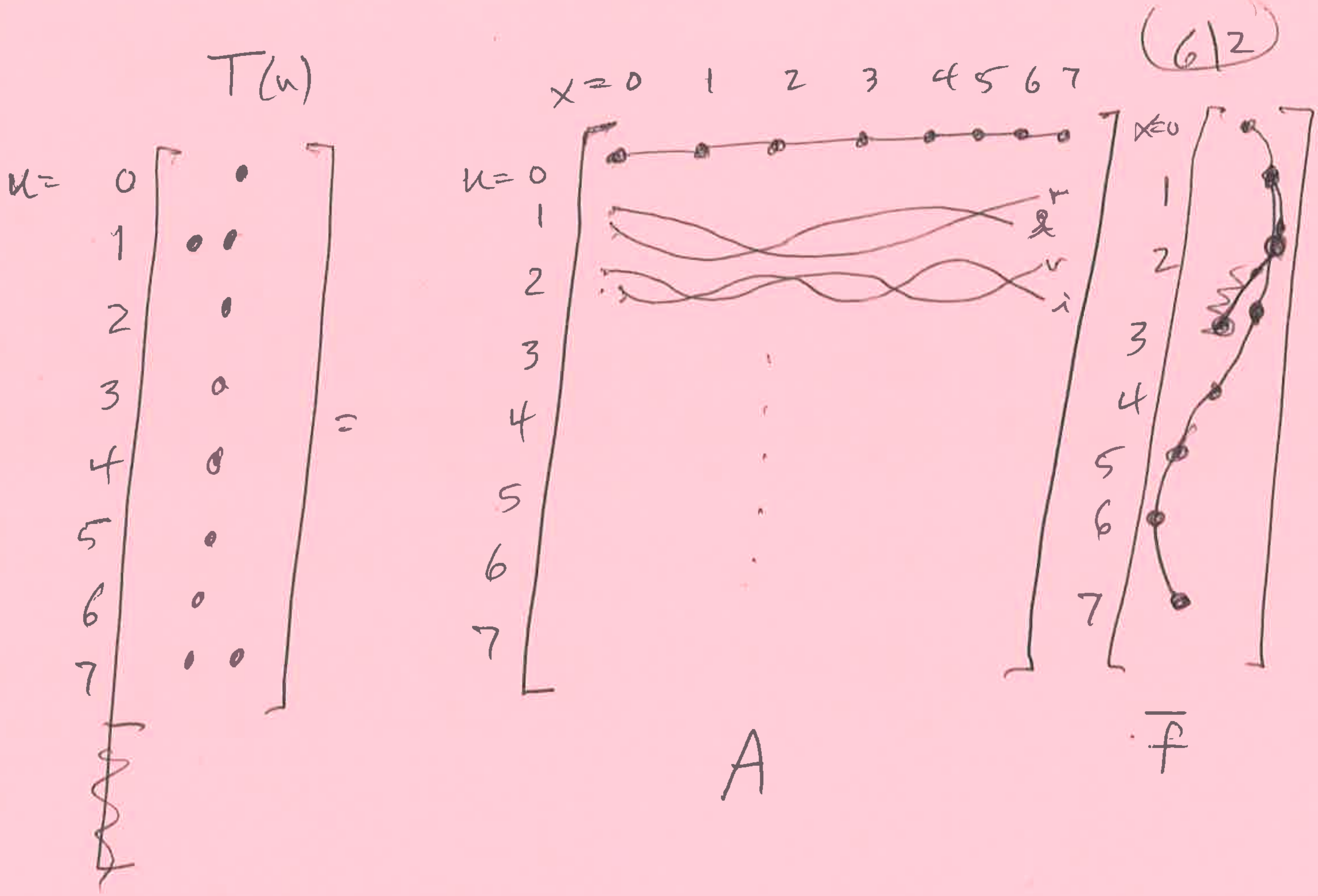
forward transform kernel

$$T(u) = \sum_{x=0}^{N-1} f(x)\, r(x,u)$$

transform variable

x: spatial variable

$T(u) = \langle f(x), r(x,u) \rangle$



See cs6640_week6

Note relation to correlation:

transform measures degree to which a function matches a selected set of basis vectors.

Consider a couple of prob. density functions:

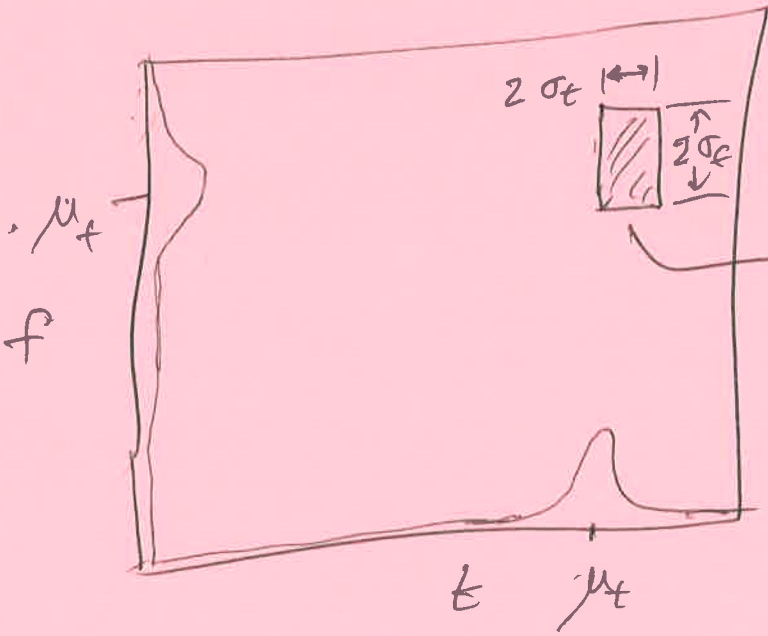
$$\cancel{p_h(t) = |h(t)|^2 / \|h(t)\|^2}$$

$$\mu_t = \frac{1}{\|\cancel{h(t)}\|^2} \int_{-\infty}^{\infty} t\,|h(t)|^2\,dt \qquad \sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t-\mu_t)^2 |h(t)|^2\,dt}{\|\cancel{h(t)}\|^2}$$

$$\cancel{p_H(f) = |H(f)|^2 / \|H(f)\|^2}$$

$$\mu_f = \frac{1}{\|\cancel{H(f)}\|^2} \int_{-\infty}^{\infty} f\,|H(f)|^2\,df \qquad \sigma_f^2 = \frac{\int_{-\infty}^{\infty} (f-\mu_f)^2 |H(f)|^2\,df}{\|\cancel{H(f)}\|^2}$$

Look at where most prob. density falls in time-frequency plane:



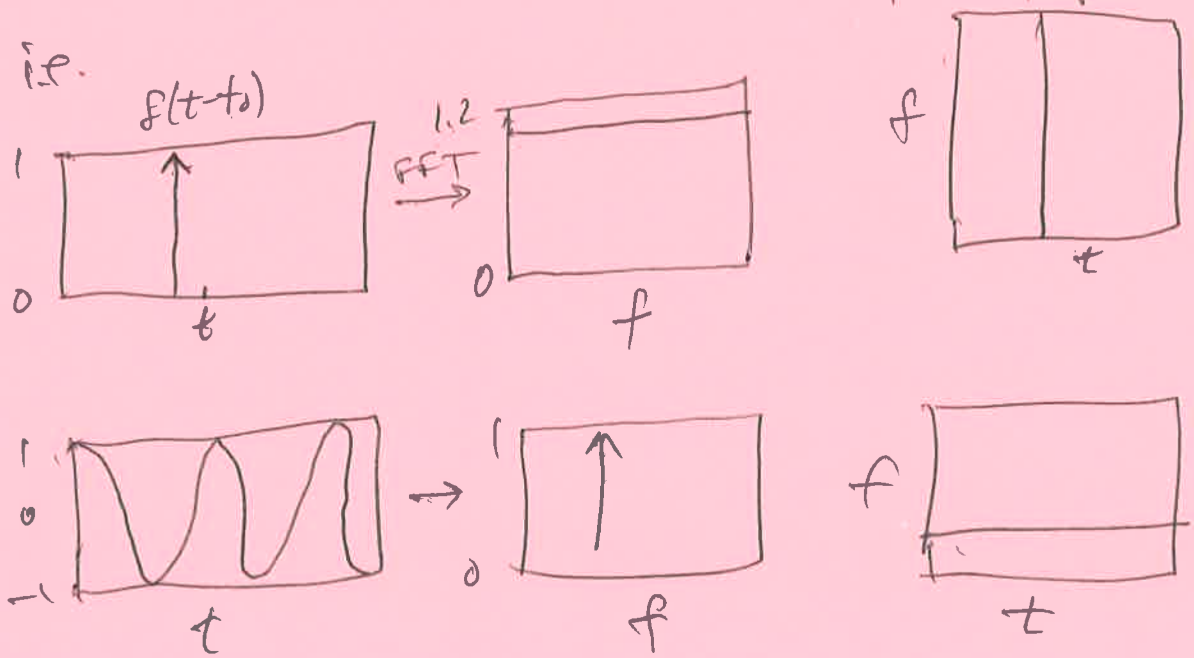
Heisenberg box of area $4\sigma_t\sigma_f$

$$\sigma_t^2\,\sigma_f^2 \geq \frac{1}{16\pi^2}$$

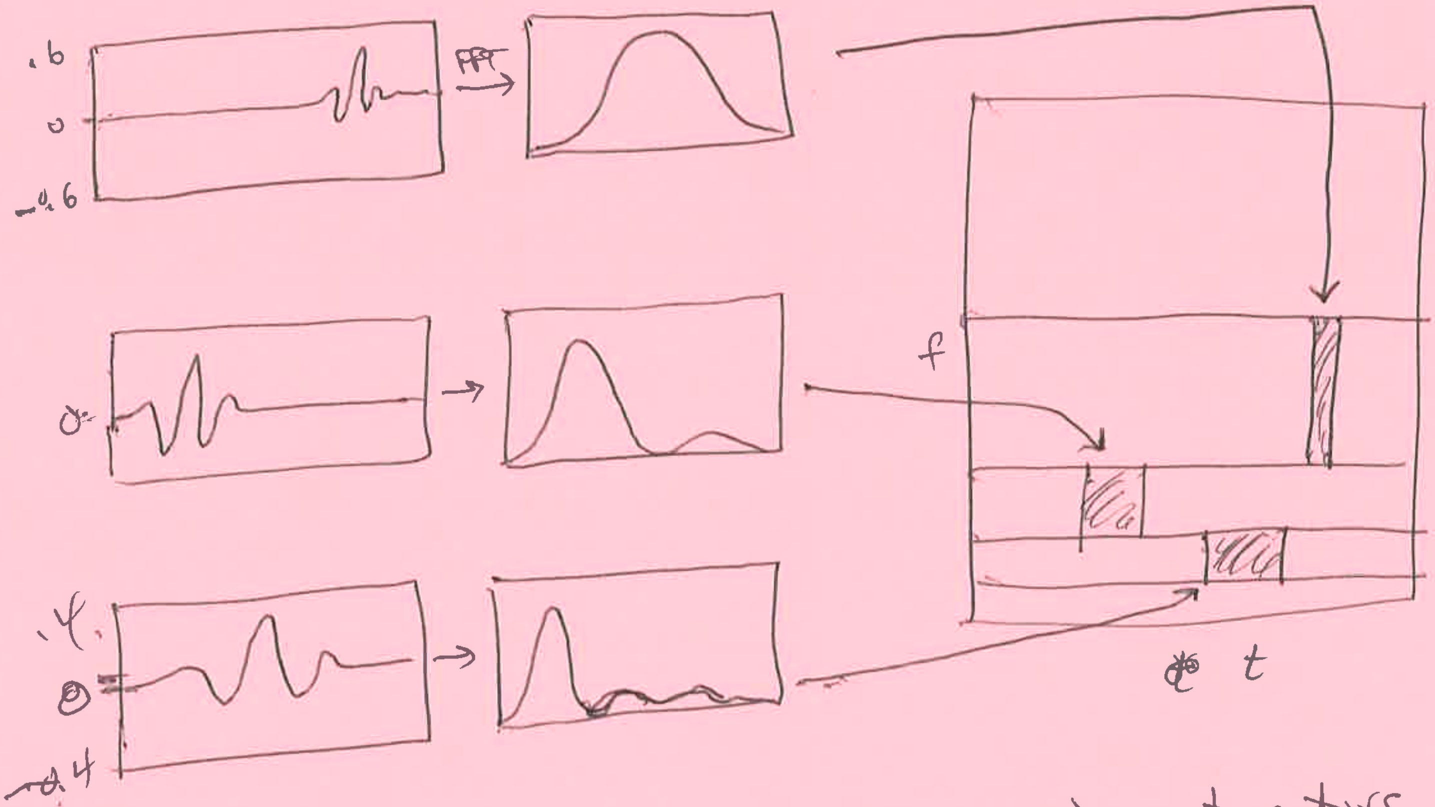
$\Rightarrow$ σ_t & σ_f cannot both be arbitrarily small.

i.e.

Time-freq plane



Some non-zero measure basis functions:



In 2-D, basis images (arrays) - not vectors

$$FT = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & -j & -j\omega & -1 & -\omega & j & j\omega \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

see CS6640 - week6

$\cancel{\omega = e^{-j2\pi/8}}$ (p. 473)

wavelets

consider set of scaled + translated functions

$$\{ \varphi_{j,k}(x) \mid j, k \in \mathbb{Z} \}$$

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k)$$

Consider Haar transform

$$h_u(x) = \begin{cases} 1 & u=0 \quad 0 \le x < 1 \\ 2^{p/2} & u>0 \text{ and } q/2^p \le x < (q+0.5)/2^p \\ -2^{p/2} & u>0 \text{ and } (q+0.5)/2^p \le x < (q+1)/2^p \\ 0 & \text{otherwise} \end{cases}$$

$u \in \mathbb{Z} \ni u>0$ + can be uniquely decomposed as

$$u = \underbrace{2^p}_{\text{largest power of 2 in } u} + q$$

$h_0(x) = 1$

$$\forall u \neq 0 \quad h(x) = \begin{cases} 2^{p/2} & \text{for } \frac{q}{2^p} \le x < \frac{q+.5}{2^p} \\ -2^{p/2} & \text{for } \frac{q+.5}{2^p} \le x < \frac{q+1}{2^p} \end{cases}$$

$p \Rightarrow$ amplitude

$q \Rightarrow$ position

wavelets obtain frequency/time info to allow analysis of time-varying signals, etc.

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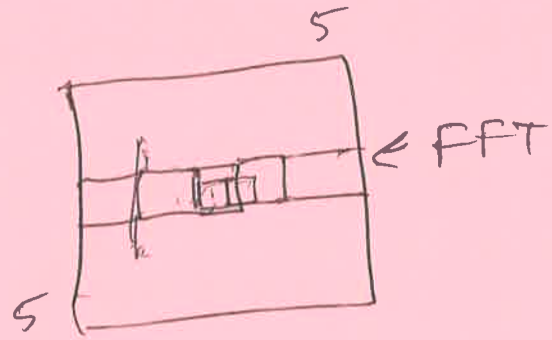
Problems (1)
FFT texture : straight forward
start at 3,3 → M-2, N-2 5x5 window

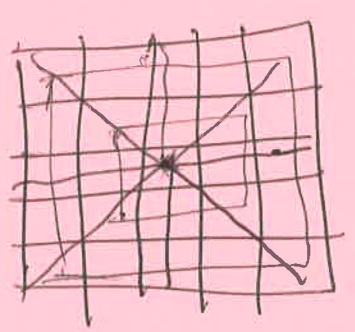
(2) radial

∀ 5x5



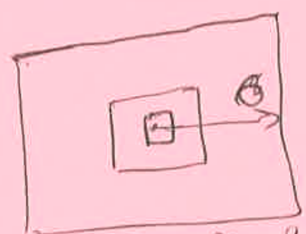
* center it? (prob. not)

pt 1



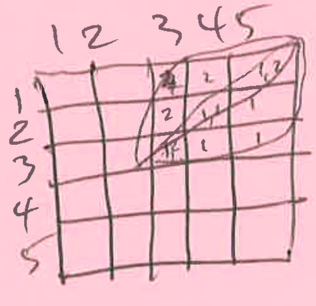
8

15 x 15 7



21 x 21 try to set up with some flexibility in order to use kmeans, get rid of borders

(3) angle



easy: set up indexes of regions

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Fourier shape

CS6640-gen-A
CS6640-gen-O

} test on these