One and Two Dimensional Fourier Analysis

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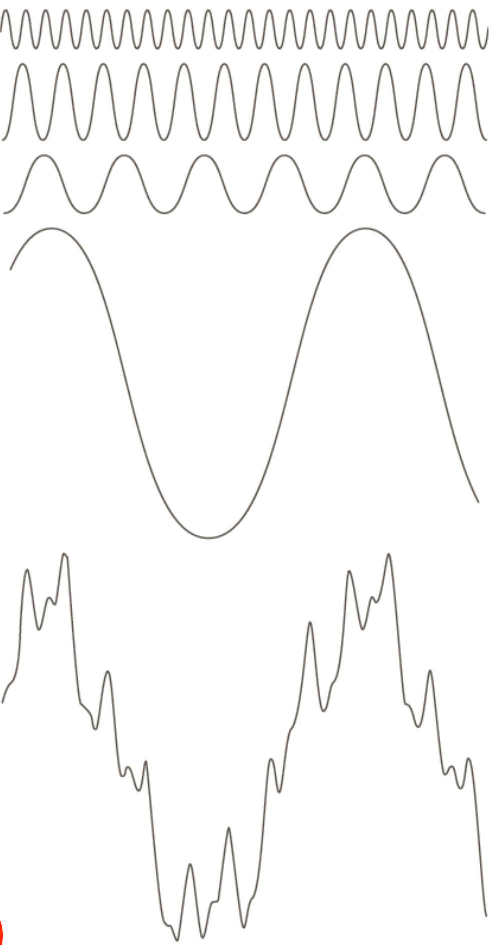
Fourier Series

- J. B. Joseph Fourier, 1807
  - Any periodic function can be expressed as a weighted sum of sines and/or cosines of different frequencies.



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

What is the period of this function?

Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$$

- f(t) periodic signal with period T
- Frequency of sines and cosines

The complex exponentials form an orthogonal basis for the range [-T/2,T/2] or any other interval with length T such as [0,T]

Types of functions

	Continuous f(t)	Discrete f(n)
Periodic	*Fourier series*	*Discrete Fourier series*
Non-periodic	*Fourier transform*	***Discrete Fourier transform***

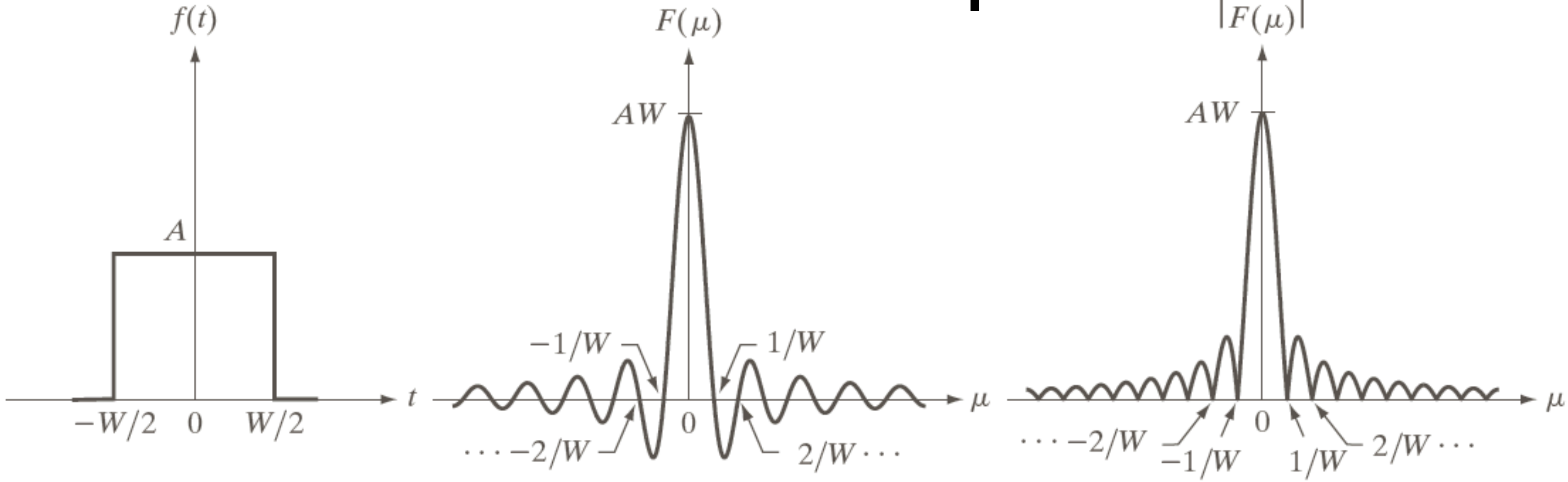
Fourier Transform Pair

$$F(\mu) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

$$f(t) = \mathcal{F}^{-1}\{F(\mu)\} = \int_{\mu=-\infty}^{\mu=\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

- The domain of the Fourier transform is the frequency domain.
  - If ***t*** is in *seconds*, ***mu*** is in *Hertz (1/seconds)*
- The function f(t) can be recovered from its Fourier transform.

Fourier Transform example



- Fourier transform of the *box* function is the *sinc* function.
- In general, the Fourier transform is a complex quantity. In this case it is real.
- The magnitude of the Fourier transform is a real quantity, called the *Fourier spectrum* (or frequency spectrum).

Convolution and Fourier Trans.

$$
\begin{aligned}
f(t) \star h(t) &= \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \\
\mathcal{F}\{f(t) \star h(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j2\pi\mu t}dt \\
&= \int_{-\infty}^{\infty} f(\tau) \underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j2\pi\mu t}dt\right]}_{\mathcal{F}\{h(t-\tau)\}=H(\mu)e^{-j2\pi\mu\tau}} d\tau
\end{aligned}
$$

Can see this by change of variables t' = t - T

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} f(\tau)H(\mu)e^{-j2\pi\mu\tau}d\tau \\
&= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi\mu\tau}d\tau = H(\mu)F(\mu)
\end{aligned}
$$

- Convolution in time domain is multiplication in frequency domain
- Multiplication in time domain is convolution in frequency domain

$$f(t) \star h(t) \iff H(\mu)F(\mu)$$

$$f(t)h(t) \iff H(\mu) \star F(\mu)$$

Unit impulse function

$$\delta(t) = \begin{cases} \infty & if \quad t = 0 \\ 0 & if \quad t \neq 0 \end{cases}$$

- Properties
  - Unit area $\int_{-\infty}^{\infty} \delta(t) dt = 1$
  - Sifting

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_o) dt = f(t_o)$$

Unit discrete impulse

- x: Discrete variable

$$\delta(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x \neq 0 \end{cases}$$

- Properties

$$\sum_{x=-\infty}^{x=\infty} \delta(x) = 1$$

$$\sum_{x=-\infty}^{x=\infty} f(x)\delta(x - x_o) = f(x_o)$$

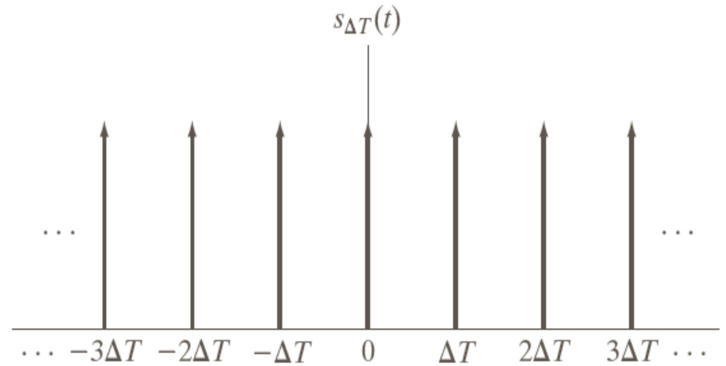
Fourier Transform of Impulses

$$\begin{aligned}
\mathcal{F}\{\delta(t)\} &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt \\
&= e^{-j2\pi\mu 0} = e^0 = 1
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}\{\delta(t - t_o)\} &= \int_{-\infty}^{\infty} \delta(t - t_o) e^{-j2\pi\mu t} dt \\
&= e^{-j2\pi\mu t_o} \\
&= \cos 2\pi\mu t_o - j \sin 2\pi\mu t_o
\end{aligned}$$

Impulse train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}t}$$

- Periodic function (period = ΔT) so can be represented as a Fourier sum

$$\begin{aligned} c_n &= \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt \\ &= \frac{1}{\Delta T} e^0 = \frac{1}{\Delta T} \end{aligned}$$

Fourier Trans. of Impulse Train

$$s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t}$$

Substitute for c_n

$$\begin{aligned}
\mathcal{F}\{s_{\Delta T}(t)\} &= \mathcal{F}\left\{ \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \right\} \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \mathcal{F}\left\{ e^{j \frac{2\pi n}{\Delta T} t} \right\} \\
&= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

Linearity of Fourier transform

Duality

FT of an impulse train is an impuse train!

$$\boxed{\begin{array}{lll} Duality: & if & \mathcal{F}\{f(t)\} \rightarrow F(\mu) \\ & then & \mathcal{F}\{F(t)\} \rightarrow f(-t) \end{array}}$$

Proof of duality for impulses

$$\mathcal{F}\{\delta(t-t_o)\} = \quad e^{-j2\pi\mu t_o}$$ From before

$$\begin{aligned}
\mathcal{F}^{-1}\{\delta(\mu-a)\} &= \int_{-\infty}^{\infty} \delta(\mu-a)e^{j2\pi\mu t}d\mu \\
&= \int_{-\infty}^{\infty} \delta(-\mu+a)e^{j2\pi\mu t}d\mu \\
&= \int_{-\infty}^{\infty} \delta(\mu'+a)e^{-j2\pi\mu' t}d\mu' \\
&= e^{j2\pi at} \\
\mathcal{F}\left\{\mathcal{F}^{-1}\{\delta(\mu-a)\}\right\} &= \mathcal{F}\{e^{j2\pi at}\} \\
\delta(\mu-a) &= \mathcal{F}\{e^{j2\pi at}\}
\end{aligned}$$

Take Fourier Trans. of both sides

Discrete Sampling and Aliasing

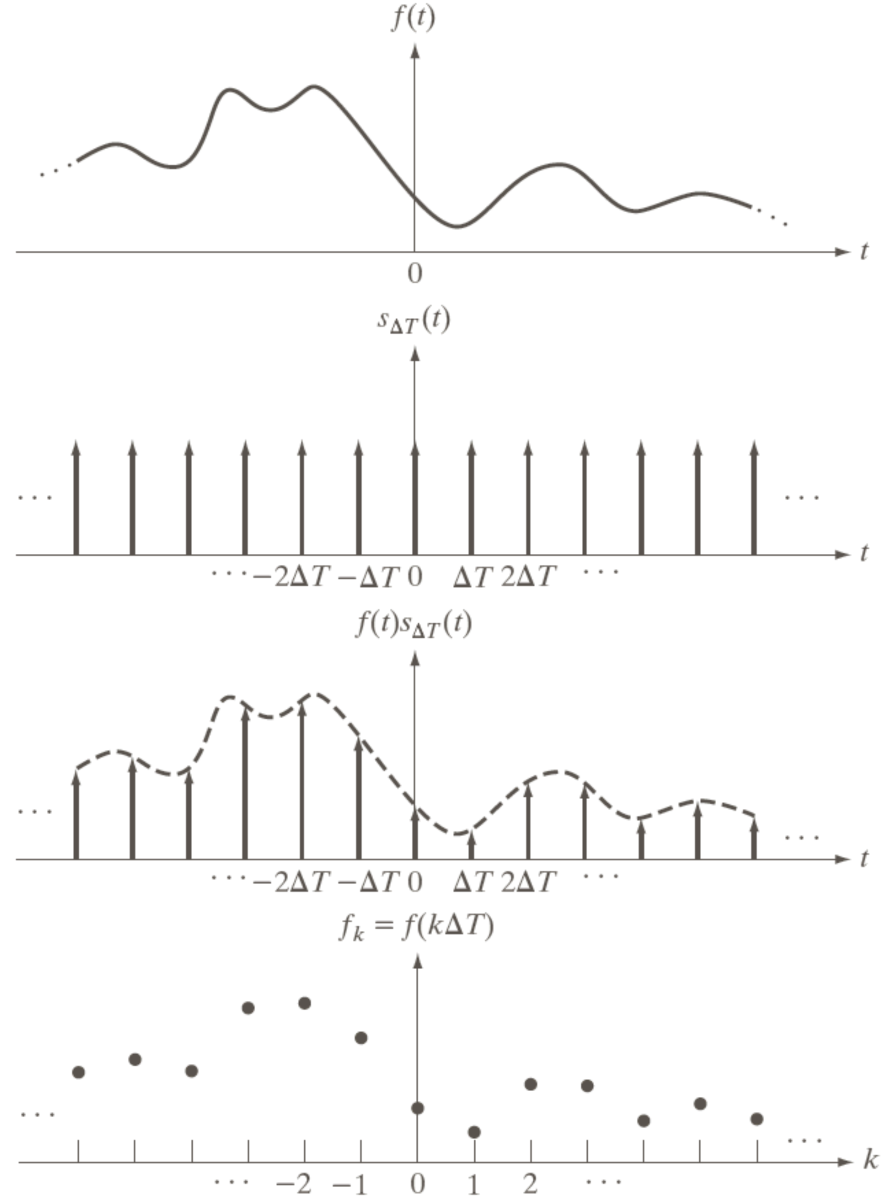
- Digital signals and images are discrete representations of the real world
  - Which is continuous
- What happens to signals/images when we sample them?
  - Can we quantify the effects?
  - Can we understand the artifacts and can we limit them?
  - Can we reconstruct the original image from the discrete data?

Sampling

- We can sample continuous function f(t) by multiplication with an impulse train

$$\begin{aligned}\tilde{f}(t) &= f(t)s_{\Delta T}(t) \\ &= \sum_{n=-\infty}^{\infty} f(t)\delta(t-n\Delta T)\end{aligned}$$

$$\begin{aligned}f_k &= f(k\Delta T) \\ &= \int_{-\infty}^{\infty} f(t)\delta(t-k\Delta T)dt\end{aligned}$$



Fourier trans. of sampled func.

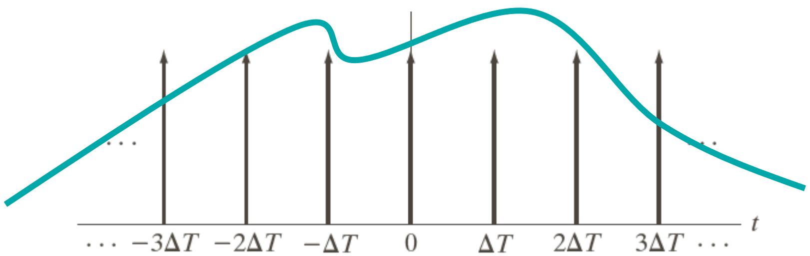
$$\begin{aligned}
\mathcal{F}\{s_{\Delta T}(t)\} &= \mathcal{F}\left\{\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{\Delta T}t}\right\} \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \mathcal{F}\left\{e^{j\frac{2\pi n}{\Delta T}t}\right\} \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

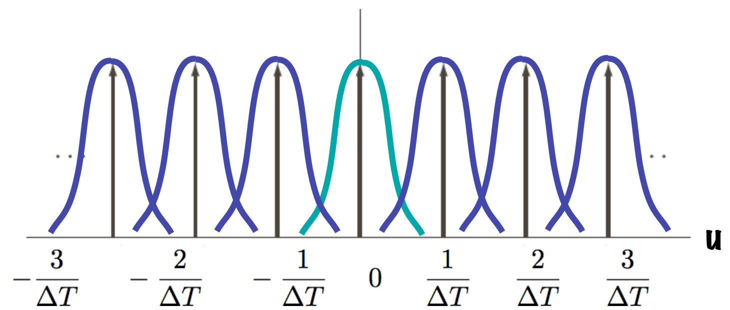
$$\begin{aligned}
\tilde{F}(\mu) &= \mathcal{F}\{f(t)s_{\Delta T}(t)\} \\
&= F(\mu) \star S(\mu) \\
&= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
&= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
\end{aligned}$$

- What does this mean?

Fourier Transform of A Discrete Sampling

$$\tilde{f}(t) = f(t)s(t) \quad \longleftrightarrow \quad \tilde{F}(u) = F(u) * S(u)$$

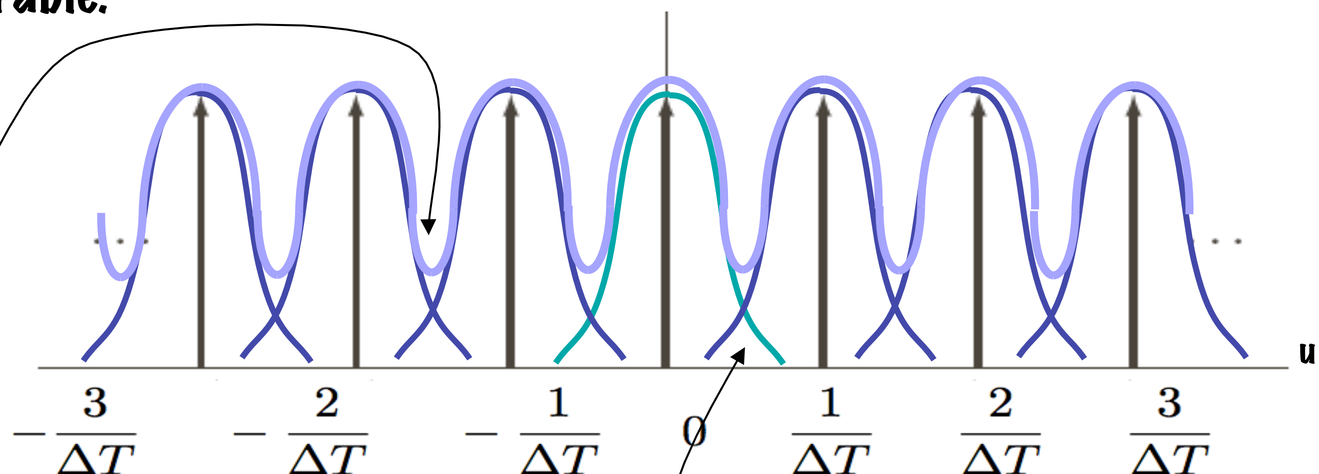




Fourier Transform of A Discrete Sampling

Frequencies get mixed. The original signal is not recoverable.

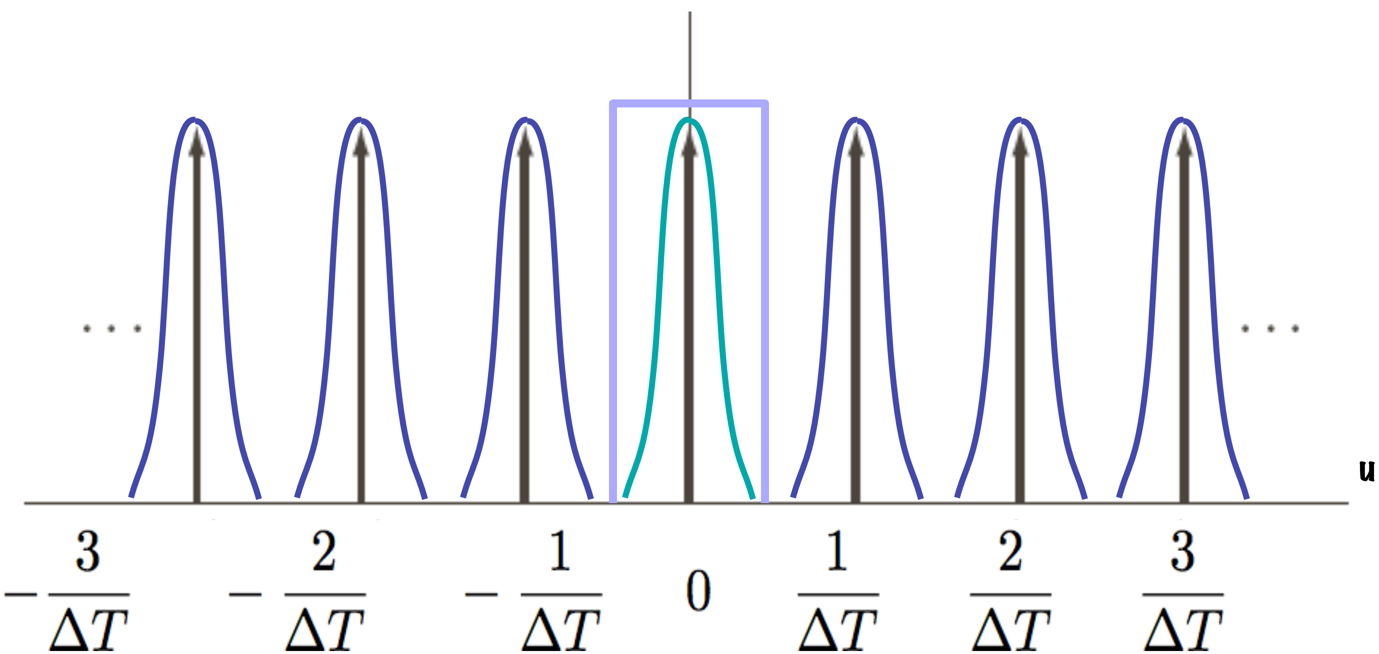
$$\tilde{F}(u) = F(u) * S(u)$$



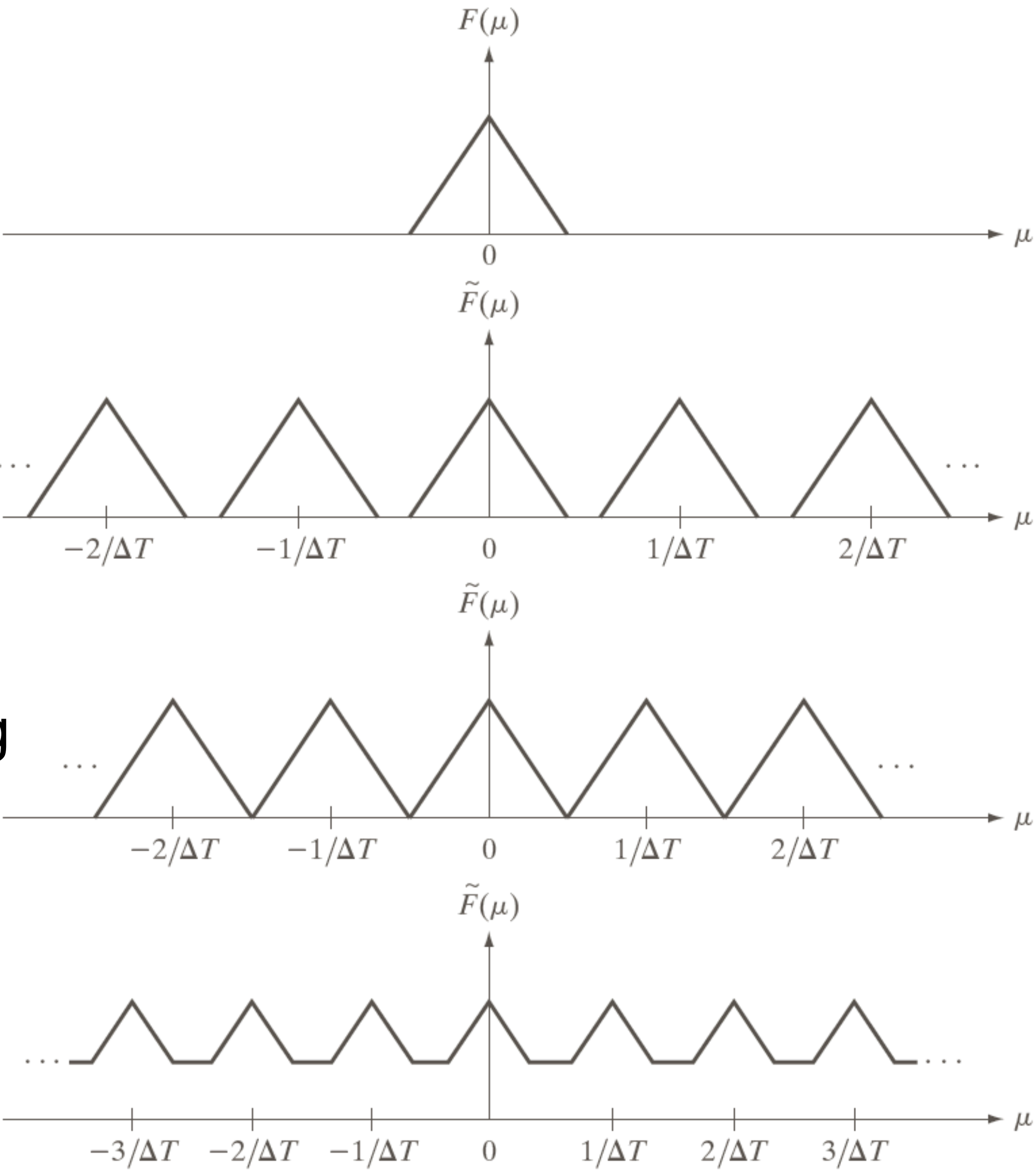
Energy from higher freqs gets folded back down into lower freqs – <u>Aliasing</u>

What if F(u) is Narrower in the Fourier Domain?

- No aliasing!
- How could we recover the original signal?



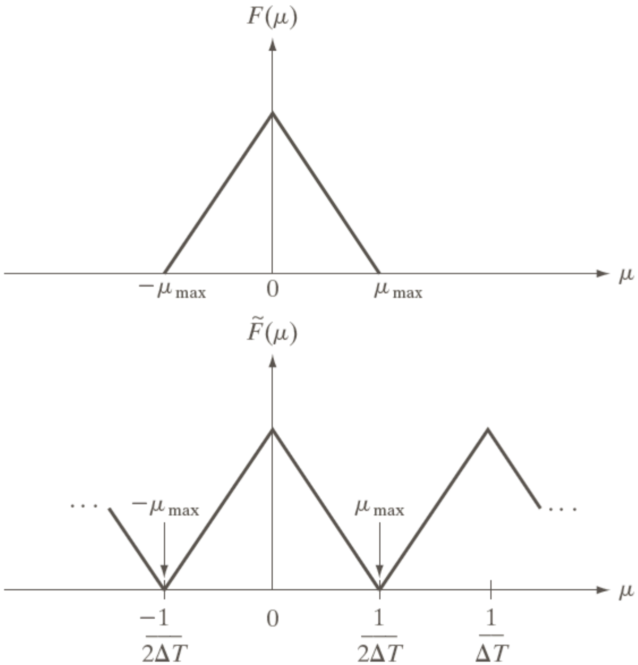
- Fourier transform of band-limited signal
- Over-sampling
- Critically-sampling
- Under-sampling



What Comes Out of This Model

- Sampling criterion for complete recovery
- An understanding of the effects of sampling
  - Aliasing and how to avoid it
- Reconstruction of signals from discrete samples

Sampling theorem



- When can we recover f(t) from its sampled version?
  - f(t) has to be band-limited
  - If we can isolate a single copy of F(μ) from the Fourier transform of the sampled signal.

$$F(\mu) = 0 \quad \forall \mu > \mu_{max}$$

$$where \ \mu_{max} < \infty$$

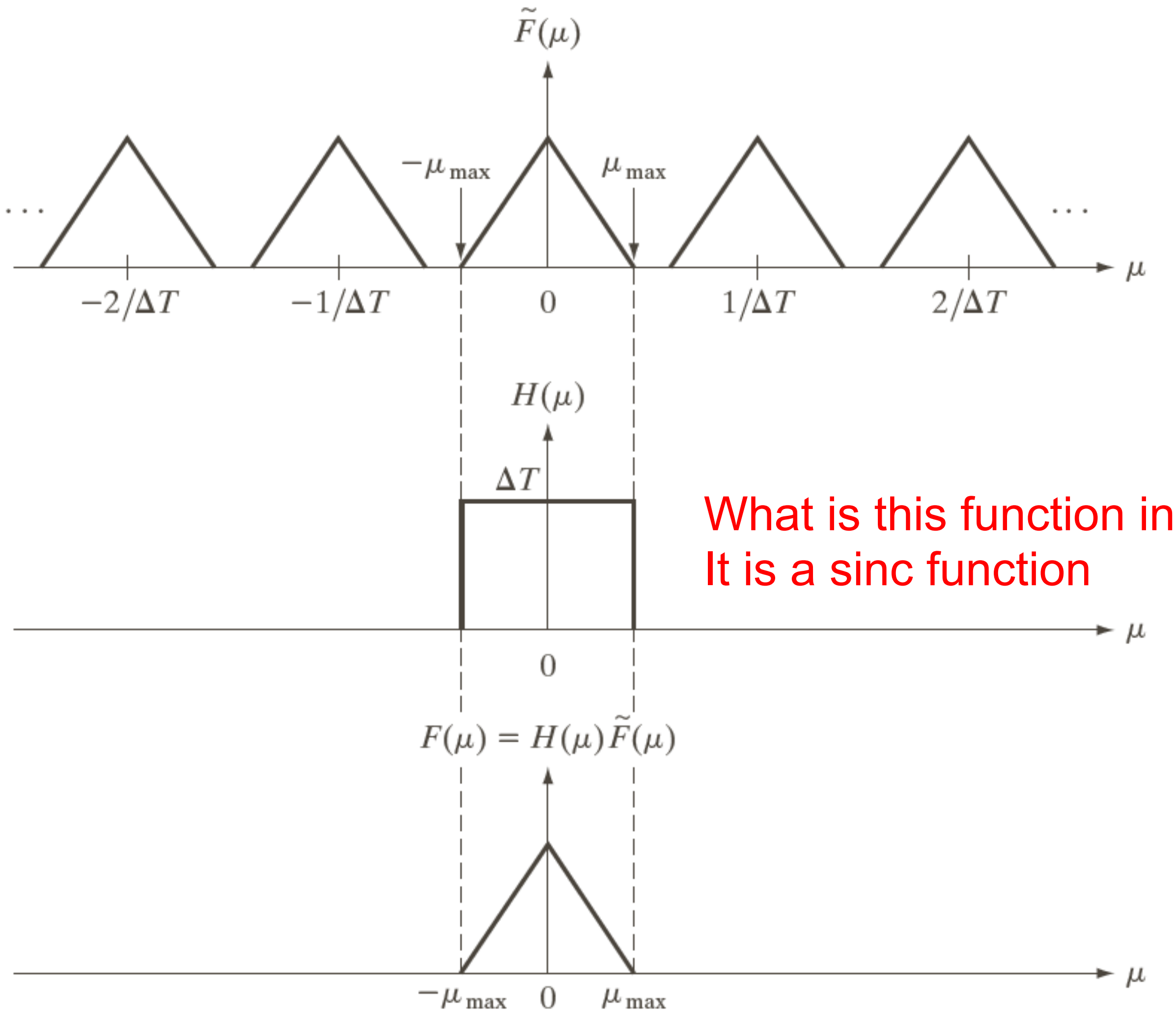
$$\frac{1}{\Delta T} > 2\mu_{max}$$

Nyquist rate

Sampling Theorem

- Quantifies the amount of information in a signal
  - Discrete signal contains limited frequencies
  - Band-limited signals contain no more information then their discrete equivalents
- Reconstruction by cutting away the repeated signals in the Fourier domain
  - Convolution with sinc function in space/time

Function recovery from sample



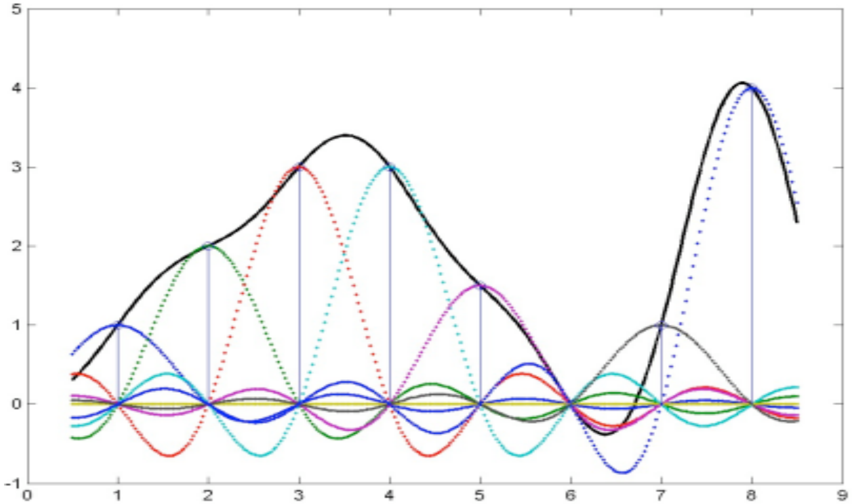
What is this function in time?
It is a sinc function

Reconstruction

- Convolution with sinc function

$$f(t) = \tilde{f}(t) * \mathbb{F}^{-1}\left[\mathrm{rect}\left(\Delta \mathrm{T} \mathrm{u}\right)\right]$$

$$= \left(\sum_k f_k \delta(t - k\Delta T)\right) * \mathrm{sinc}\left(\frac{\mathrm{t}}{\Delta \mathrm{T}}\right) = \sum_{\mathrm{k}} \mathrm{f_k}\ \mathrm{sinc}\left(\frac{\mathrm{t} - \mathrm{k}\Delta\mathrm{T}}{\Delta\mathrm{T}}\right)$$



Note: Sinc function has infinite duration. Why?
Ideal reconstruction is not feasible in practice
What happens if you truncate the sinc?

Aliasing example

$$f(t) = \sin(\pi t)$$

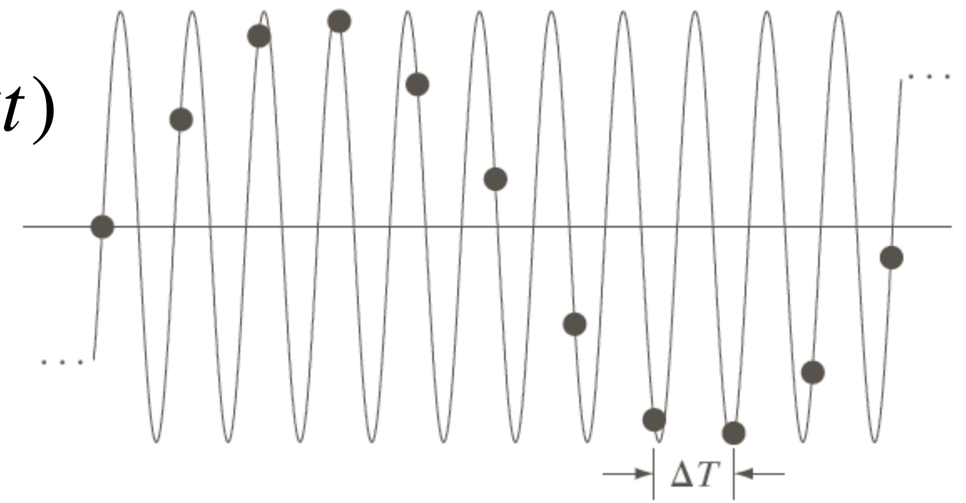


Figure: Sampling rate less than Nyquist rate

Period = 2, Frequency = 0.5
Nyquist rate = 2 x 0.5 = 1

Sampling rate must be strictly greater than the Nyquist rate. What happens if we sample this signal at exactly the Nyquist rate?

Inevitable aliasing

- No function of finite duration can be band-limited!!
- Assume we have a band-limited signal of infinite duration. We limit the duration by multiplication with a box function:
  - We already know the Fourier transform of the box function is a sinc function in frequency domain which extends to infinity.
  - Multiplication in time domain is convolution in frequency domain. Therefore, we destroyed the band-limited property of the original signal

Two-dimensional Fourier Transform Pair

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}dtdz$$

$$f(t,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mu,\nu)e^{j2\pi(\mu t+\nu z)}d\mu d\nu$$

Properties from 1D carry over to 2D:
Shifting in space <-> Multiplication with a complex exponential
Duality of multiplication and convolution
Etc..

2D impulse function

$$\delta(t,z) = \begin{cases} \infty & if \quad t = z = 0 \\ 0 & if \quad otherwise \end{cases}$$

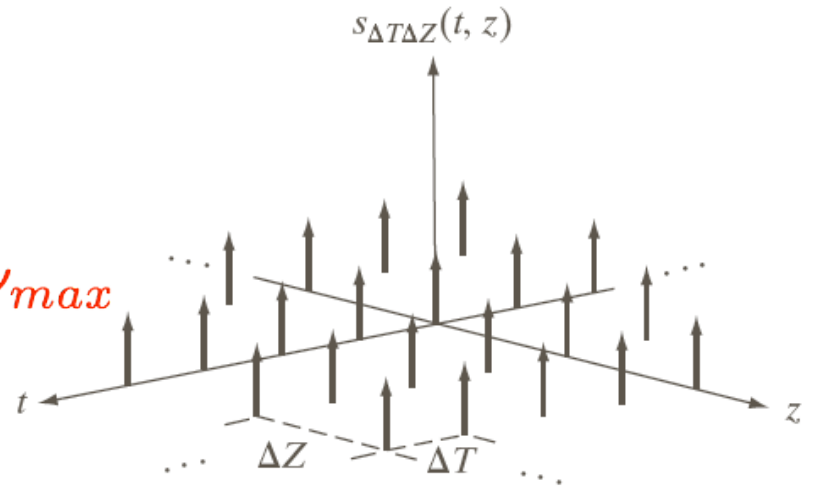
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,z)\,dt\,dz = 1$$

$$\int_{-\infty}^{\infty} f(t,z)\delta(t-t_o, z-z_o)dt = f(t_o, z_o)$$

2D sampling

- 2D impulse train as sampling function

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$



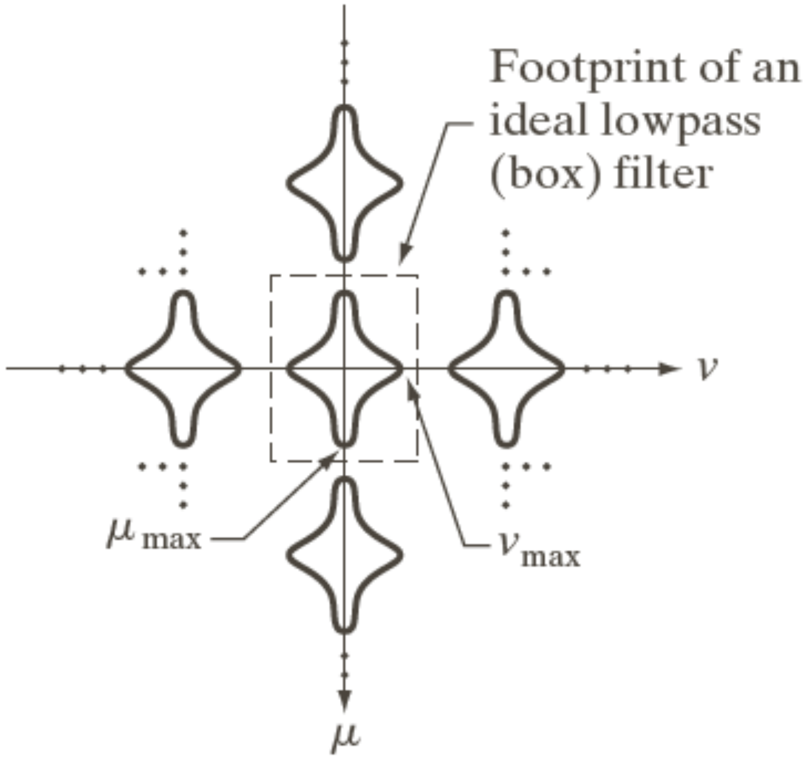
- Sampling theorem
  - Band-limited

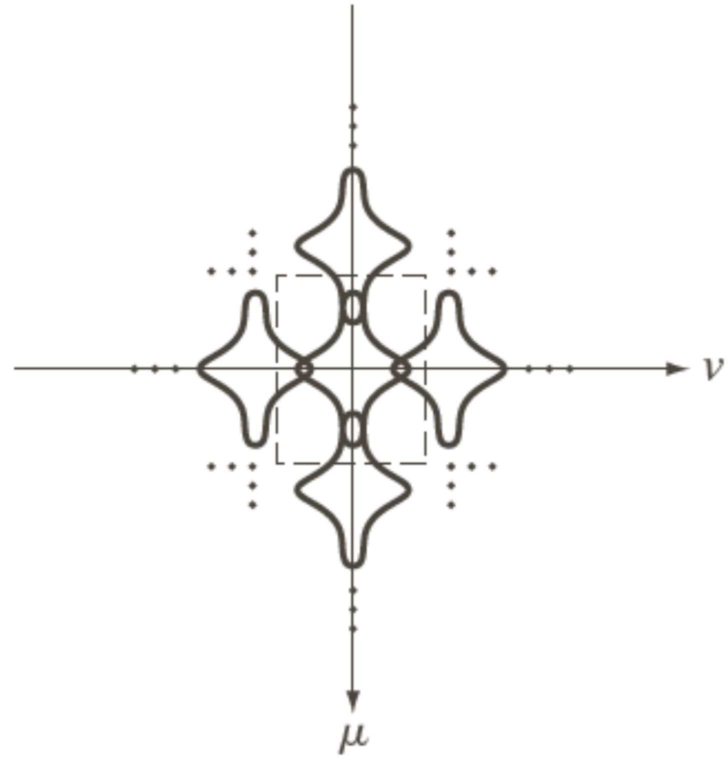
$$F(\mu, \nu) = 0 \ for \ \mu > \mu_{max} \ or \ \nu > \nu_{max}$$

  - Sampling rate limits

$$\frac{1}{\Delta T} > 2\mu_{max} \qquad \frac{1}{\Delta Z} > 2\nu_{max}$$

Aliasing in images





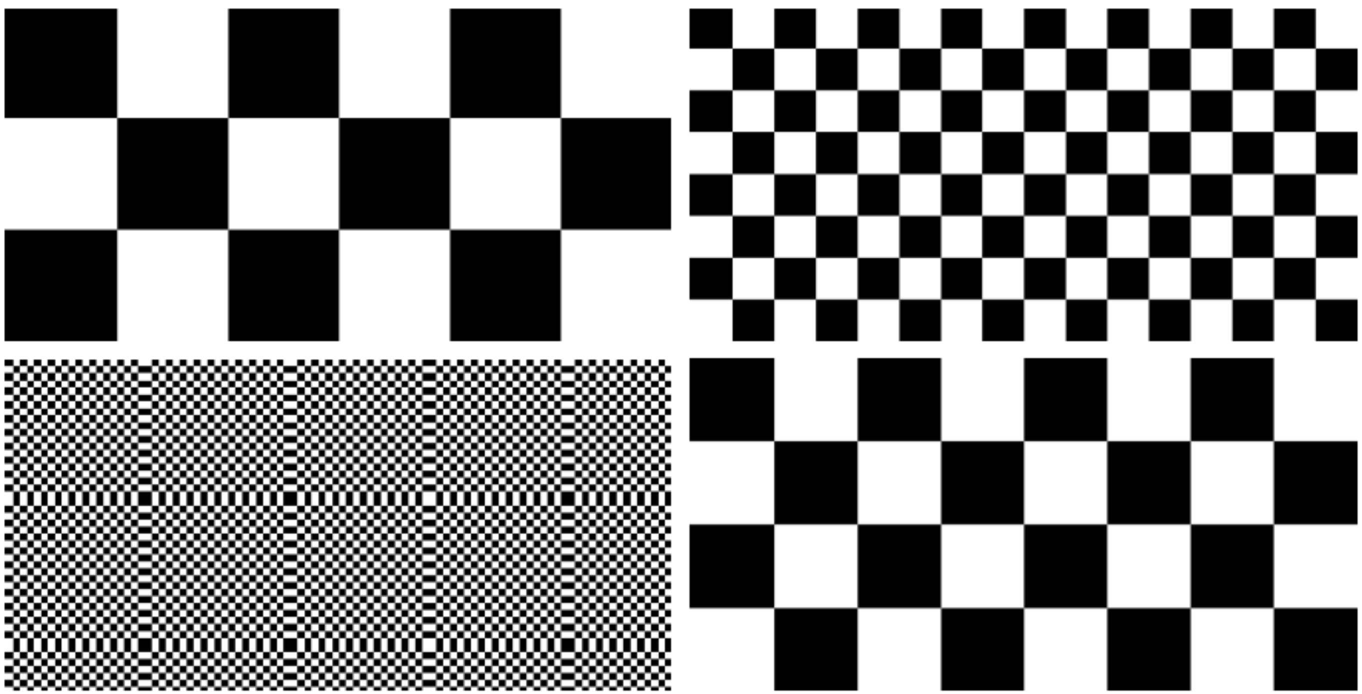
Over-sampled

Under-sampled
Aliasing

Aliasing example

- Digitizing a checkerboard pattern with 96 x 96 sample array.
  - We can resolve squares that have physical sides one pixel long or longer

16 pixels



8 pixels

0.9174 pixels

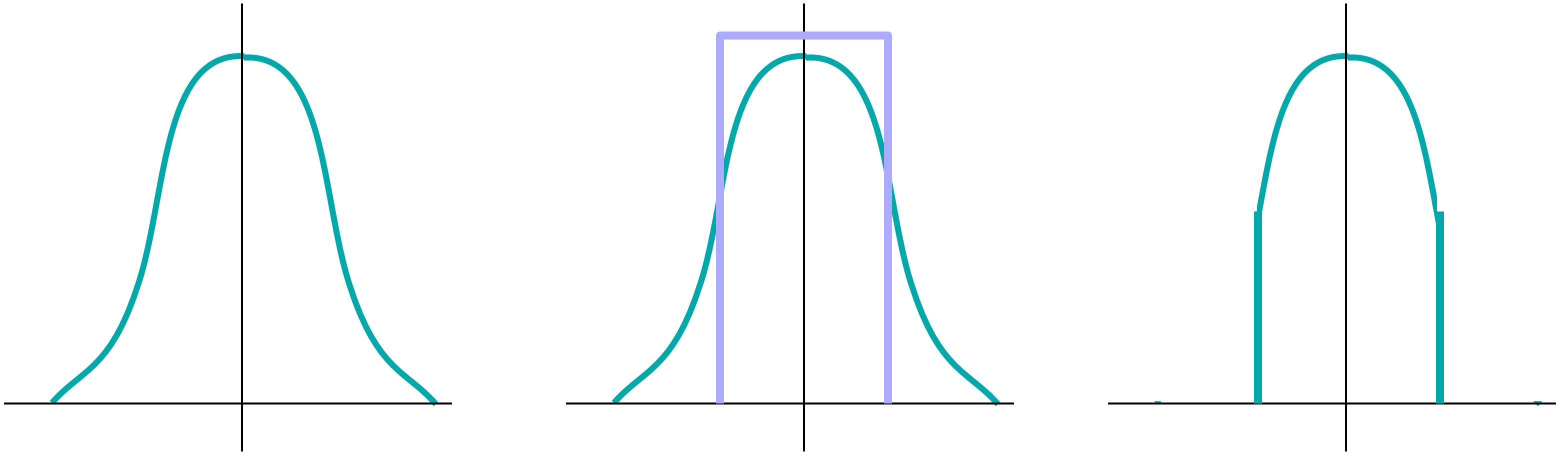
0.4798 pixels

Aliasing in images

- No time or space limited signal can be band limited
- Images always have finite extent (duration) so aliasing is always present
- Effects of aliasing can be reduced by slightly defocusing the scene to be digitized (blurring continuous signal)
- Resampling a digital image can also cause aliasing.
  - Blurring (averaging) helps reduce these effects

Overcoming Aliasing

- Filter data prior to sampling
  – Ideally - band limit the data (conv with sinc function)
  – In practice - limit effects with fuzzy/soft low pass



Overcoming alising due to image resampling

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Discrete Fourier Transform

- Fourier transform of sampled data was derived in terms of the transform of the original function:

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

- We want an expression in terms of the sampled function itself. From the definition of the Fourier Transform:

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt$$

$$
\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}
\end{aligned}
$$

$$
\begin{aligned}
f_k &= f(k\Delta T) \\
&= \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T) dt
\end{aligned}
$$

Discrete Fourier Trans. (DFT)

- Notice that the Fourier transform of the discrete signal f_n is continuous and periodic! What is the period?
- We only need to **sample one period** of the Fourier transform. This is the DFT:
  - Samples taken at $\mu = \dfrac{m}{M\Delta T}$
  - m=0,1,...,M-1

$$\boxed{F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}}$$

Discrete Fourier Transform Pair

Discrete signal $f_0, \ldots, f_{M-1}$

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad \text{m=0,1,...,M-1}$$

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad \text{n=0,1,...,M-1}$$

2D Discrete Fourier Transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

- **Notation**: From now on we will use x,y and u,v to denote discrete variables.
- f(x,y) is a M x N digital image
- F(u,v) is also a 2D matrix of size M x N. Its elements are complex quantities.

Spatial and frequency intervals

- The entire range of frequencies spanned by the DFT is

$$u \in \left[0, \frac{1}{\Delta T}\right] \; and \; v \in \left[0, \frac{1}{\Delta Z}\right]$$

- The relationship between the spatial and frequency intervals is

$$\Delta u = \frac{1}{M \Delta T} \; and \; \Delta v = \frac{1}{N \Delta Z}$$

Periodicity of DFT and 2D DFT

$$
\begin{aligned}
F(u+kM) &= \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u+kM)x/M} \\
&= \left(\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}\right) e^{-j2\pi kx} \\
&= F(u)
\end{aligned}
$$

- Above result holds because k and x are integers. This also implies f(x) obtained by the inverse DFT is periodic! For 2D:
  - $F(u, v) = F(u + k_1M , v + k_2N)$
  - $f(x, y) = f(x + k_1M , y + k_2N)$
  - k_1 and k_2 integers

Fourier spectrum and phase

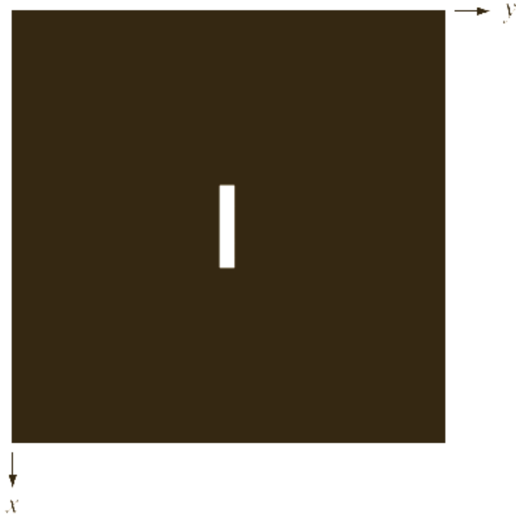
- Since the DFT is a complex quantity it can also be expressed in polar coordinates:

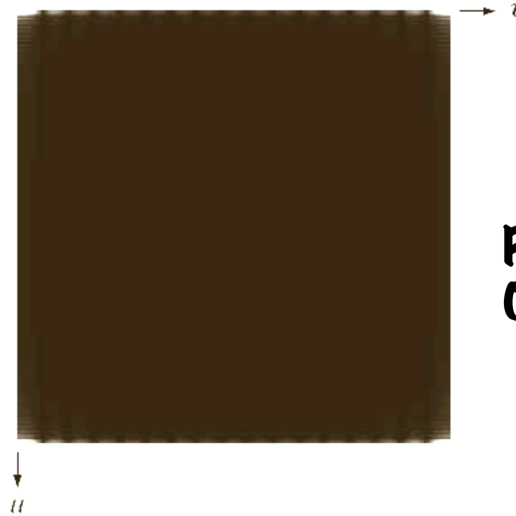
$$
\begin{aligned}
F(u,v) &= R(u,v) + jI(u,v) \\
F(u,v) &= |F(u,v)|e^{j\phi(u,v)} \\
|F(u,v)| &= \sqrt{R^2(u,v) + I^2(u,v)} \\
\phi(u,v) &= \arctan\left[\frac{I(u,v)}{R(u,v)}\right]
\end{aligned}
$$

4-quadrant arctangent, ***atan2*** command in **MATLAB**

Fourier Spectrum

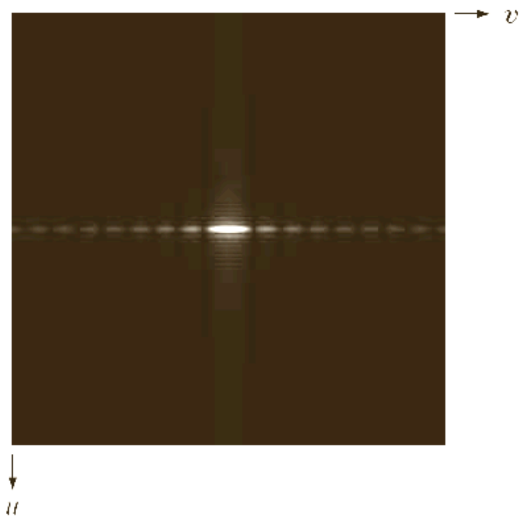
Image

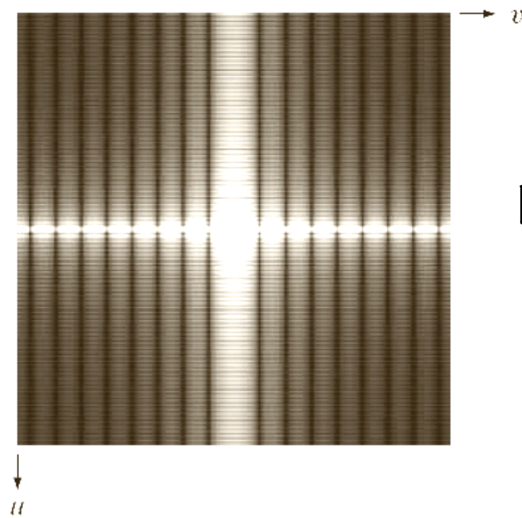




Fourier spectrum
Origin in corners

Retiled with origin
In center





Log of spectrum

Translation properties

- Translation in space

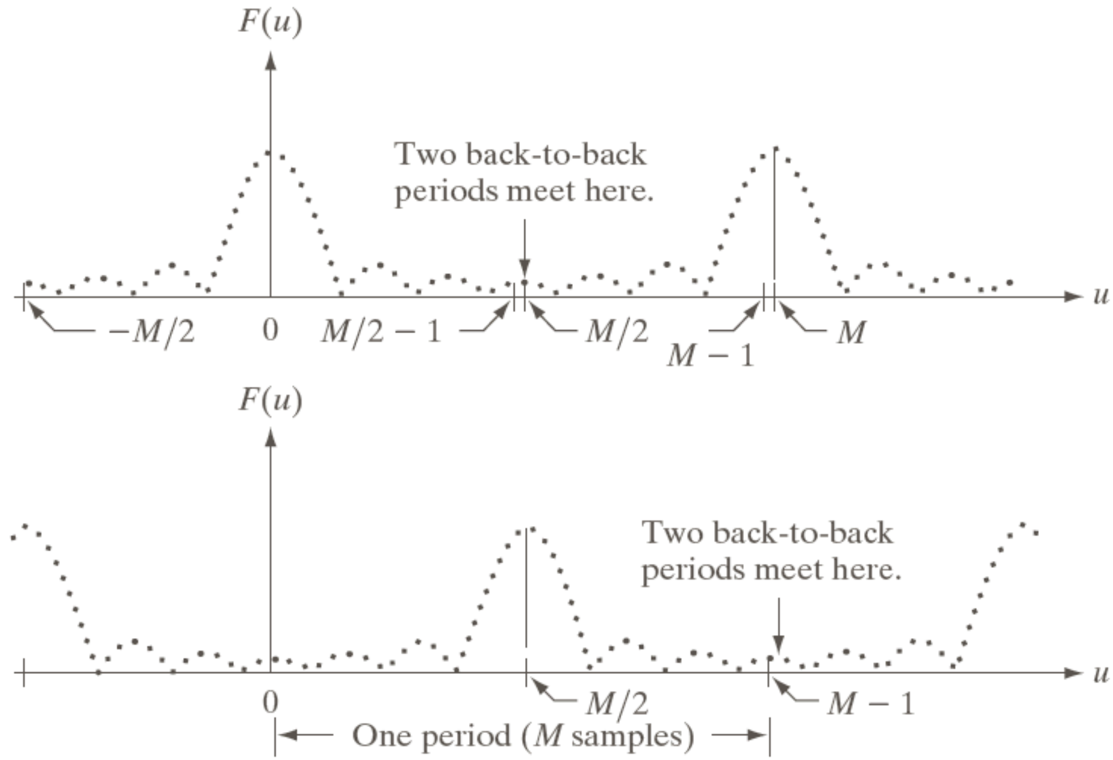
$$f(x - x_o, y - y_o) \Longleftrightarrow F(u, v) e^{-j2\pi\left(\frac{x_o u}{M} + \frac{y_o v}{N}\right)}$$

- Translation in frequency

$$f(x, y) e^{j2\pi\left(\frac{x u_o}{M} + \frac{y v_o}{N}\right)} \Longleftrightarrow F(u - u_o, v - v_o)$$

Note: Centering the Fourier transform is a shift in frequency with u_0 = M/2 and v_0 = N/2 which is a multiplication by $(-1)^{x+y}$ in space

Centering the DFT



We want half period (M/2) shift in the frequency domain:

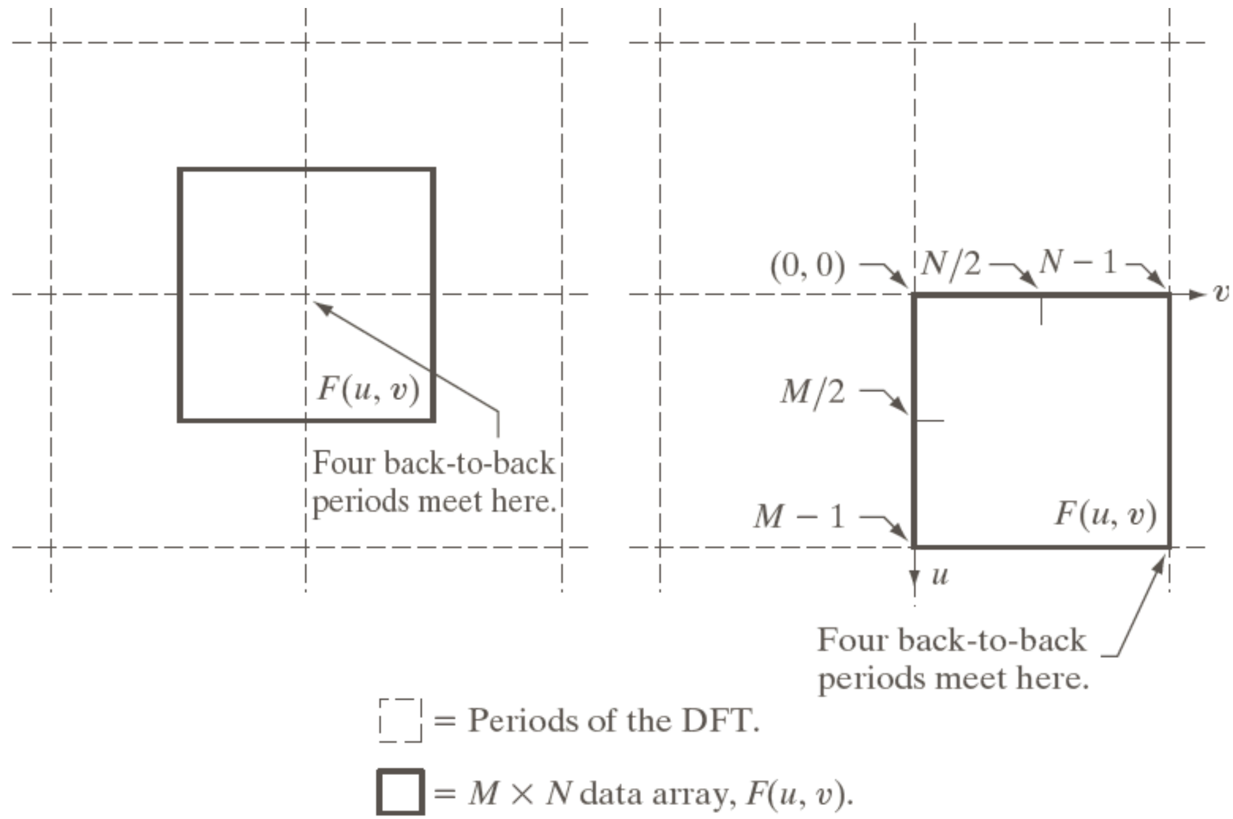
$$f(x)e^{j2\pi\frac{(M/2)x}{M}} \Leftrightarrow F(u - M/2)$$

$$f(x)e^{j\pi x} \Leftrightarrow F(u - M/2)$$

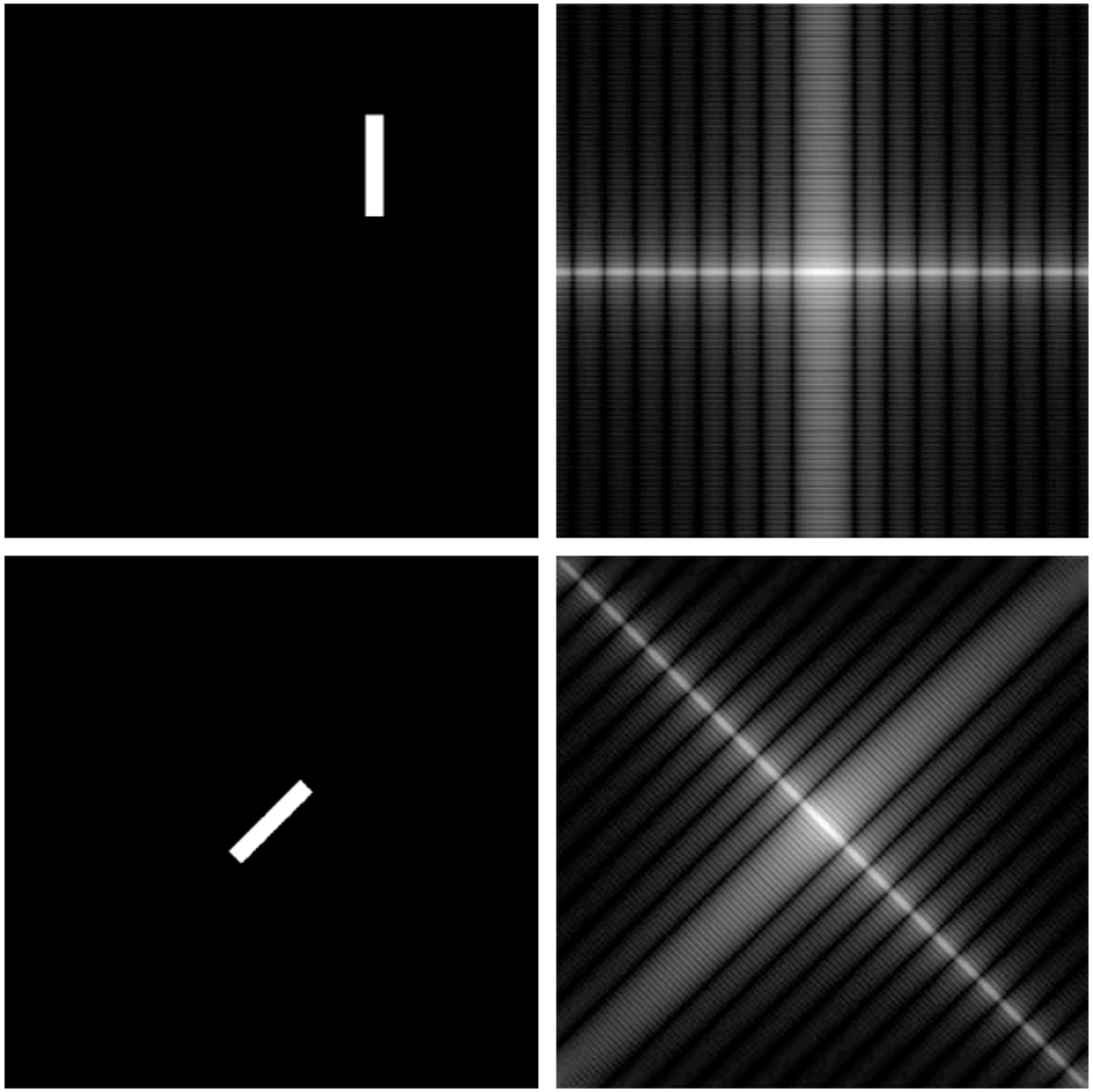
$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

In 2D...

$$f(x,y)(-1)^{x+y} \Longleftrightarrow F(u - M/2, v - N/2)$$



Translation and rotation



- Translation in space only effects the phase but not the spectrum of the DFT
- Rotation in space rotates the DFT (and hence the spectrum) by the same angle

Phase information

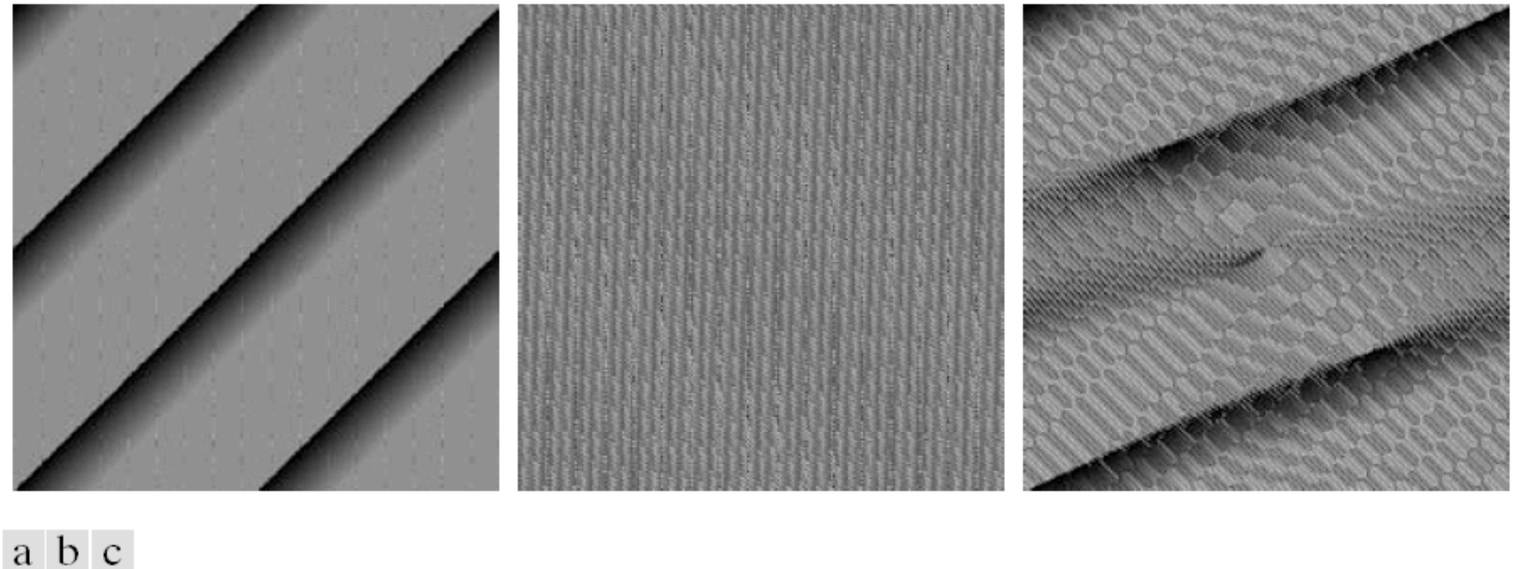


FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

- Phase angle is not intuitive, but it is critical. It determines how the various frequency sinusoids add up. This gives result to shape!

Importance of phase angle

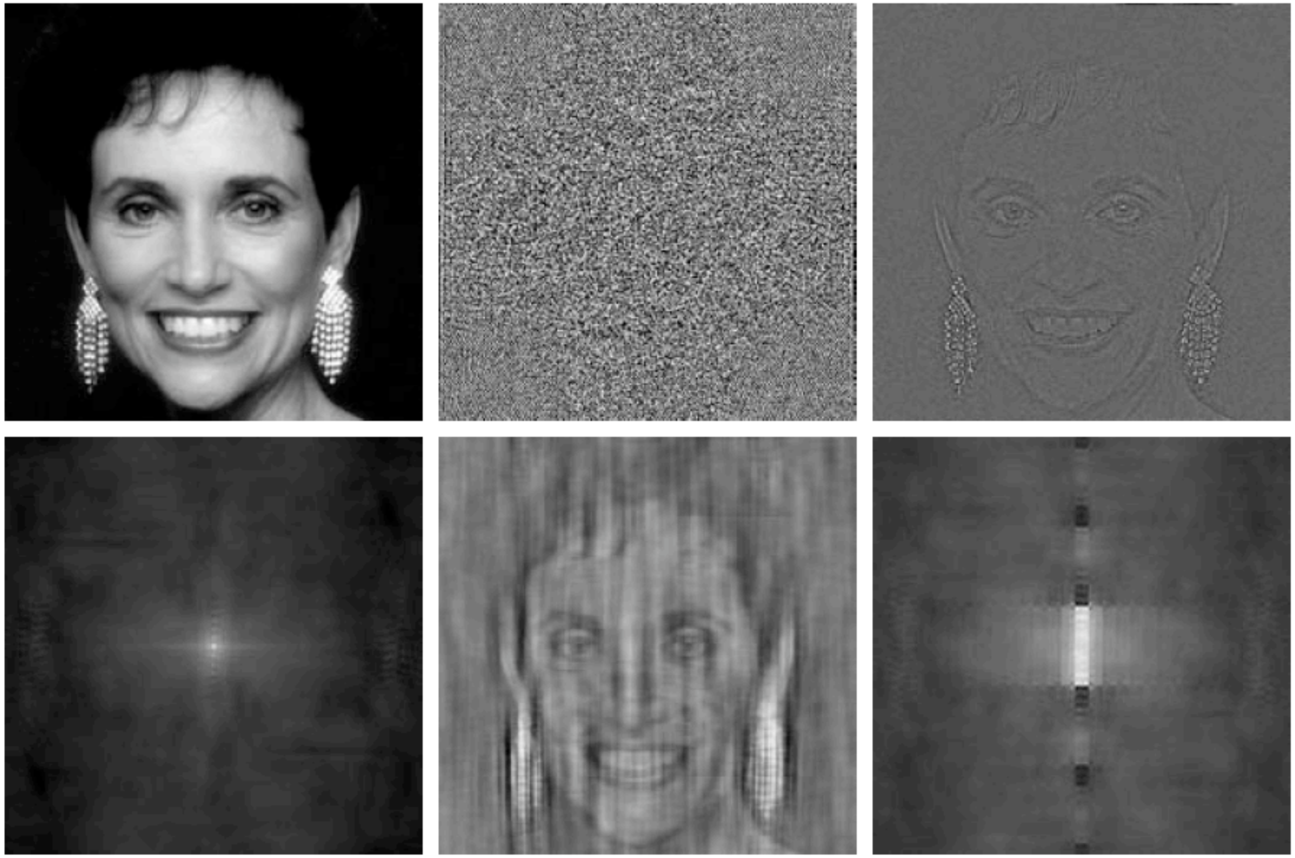


FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.