Chapter 11

deformable models

active contours : move to fit shape

snakes

(*) parametric curve

e.g., implicit specification of circle:

$$x^2 + y^2 = 4 \quad \equiv \quad \{(x,y) \ni x^2+y^2=4\}$$

$$(x-x_0)^2 + (y-y_0)^2 = 4$$

parametric specification

$$x(\theta) = x_0 + 2\cos\theta$$

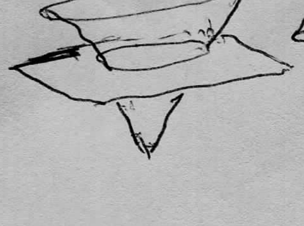
$$y(\theta) = y_0 + 2\sin\theta$$

↑ parameter: $\underline{\underline{\theta}}$

(*) seek boundary, guided by

+ internal force : {curvature, shape model}
+ external force : image features like edges

level sets : implicit representation
intersects 3D surface with plane



parametric curve

$$(x, y) = (g(s), h(s))$$

$$\left.\begin{array}{l} x(s) = g(s) \\ y(s) = h(s) \end{array}\right\} \text{parametric equation}$$

Can also use cartesian coordinates

$$\bar{c}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \pm\sqrt{r^2 - x^2} \end{bmatrix} \quad -r \leq x \leq r$$

parametric

$$\bar{c}(s) = \begin{bmatrix} r\cos(s) \\ r\sin(s) \end{bmatrix} \quad 0 \leq s \leq 2\pi$$

Snake equation

$$\bar{c}(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} \quad 0 \leq s \leq 1$$

energy:

$$E(\bar{c}) = E_{internal} + E_{external}$$

best curve $\bar{c}$ yields lowest energy

$$E_{elastic} = \frac{1}{2} \int_0^1 \alpha \left\| \frac{\partial \bar{c}(s)}{\partial s} \right\|^2 ds$$

minimized at least stretch state

$$E_{bending} = \frac{1}{2} \int_0^1 \beta \left\| \frac{\partial^2 \bar{c}(s)}{\partial s^2} \right\|^2 ds$$

smoothness of curve

$$E_{internal} = \frac{1}{2} \int \left(\alpha \| \bar{c}'(s) \|^2 + \beta \| \bar{c}''(s) \|^2 \right) ds$$

$E_{image}(\bar{c}(s))$ image-based energy functn

$$E_{external} = \int_0^1 E_{image}(\bar{c}(s))\, ds$$

Total Energy

$$E(\bar{c}(s)) = \frac{\alpha}{2} \int_0^1 \| \bar{c}'(s) \|^2 ds + \frac{\beta}{2} \int_0^1 \| \bar{c}''(s) \|^2 ds + \int_0^1 E_{image}(\bar{c}(s))\, ds$$

Minimum of E satisfies

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \bar{c}'} \right) - \frac{d^2}{ds^2}\left(\frac{\partial F}{\partial \bar{c}''} \right) - \frac{\partial F}{\partial \bar{c}} = 0$$

where $F = \frac{\alpha}{2} \| \bar{c}'(s) \|^2 + \frac{\beta}{2} \| \bar{c}''(s) \|^2 + E_{image}(\bar{c}(s))$

~~do over~~ 1st 11|3a

$$\frac{1}{2}\alpha\int_0^1 \left\| \frac{\partial \bar{c}(s)}{\partial s} \right\|^2 ds$$

$$\frac{1}{2}\alpha\int_0^1 r^2 ds = \frac{1}{2}\alpha r^2$$ $\Rightarrow$ minimal for smaller r

$$\hat{c}(s) = \begin{bmatrix} r\cos(s) \\ r\sin(s) \end{bmatrix}$$

$$\frac{\partial \bar{c}(s)}{\partial s} = \begin{bmatrix} -r\sin(s) \\ r\cos(s) \end{bmatrix}$$

$$\left\| \begin{bmatrix} -r\sin(s) \\ r\cos(s) \end{bmatrix} \right\| = \sqrt{(-r\sin(s))^2 + (r\cos(s))^2}$$

$$= \sqrt{r^2\sin^2(s) + r^2\cos^2(s)}$$

$$= \sqrt{r^2(\sin^2(s) + \cos^2(s))}$$

$$= \sqrt{r^2}$$

$$= r$$

$$\Rightarrow \left\| \begin{bmatrix} -r\sin(s) \\ r\cos(s) \end{bmatrix} \right\|^2 = r^2$$

11\4

Solving minimal energy when:

$$\alpha \bar{c}'' - \beta \bar{c}'''' - \frac{\partial E_{image}}{\partial \bar{c}} = 0$$

but $\frac{\partial E_{image}}{\partial \bar{c}}$ is gradient $\nabla E_{image} = -F$

$$\boxed{\text{So: } \alpha \bar{c}'' - \beta \bar{c}'''' + \bar{F} = 0}$$ at equilibrium

Dynamic, so add time variable

$$\frac{\partial \bar{c}(s,t)}{\partial t} = \alpha \bar{c}''(s,t) - \beta \bar{c}''''(s,t) + \bar{F}(\bar{c}(s,t))$$

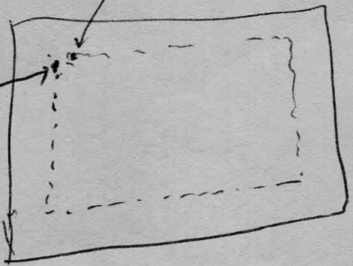
$$\frac{\partial^2 \bar{x}(t)}{\partial s^2} = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -2 & 1 & \cdots & & & 0 \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ 1 & 0 & & \cdots & & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x(0,t) \\ x(1,t) \\ \vdots \\ \vdots \\ x(K-1,t) \end{bmatrix} = D_2 \bar{x}(t)$$

$$\frac{\partial^4 \bar{x}(t)}{\partial s^4} = \begin{bmatrix} 6 & -4 & 1 & 0 & \cdots & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & \cdots & & 0 & 1 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ & & & & & & & \\ -4 & 1 & 0 & \cdots & & 0 & 1 & -4 & 6 \end{bmatrix} \begin{bmatrix} x(0,t) \\ x(1,t) \\ \vdots \\ \vdots \\ x(K-1,t) \end{bmatrix} = D_4 \bar{x}(t)$$

11\5

$$\bar{x}(t) = \begin{bmatrix} x(0,t) \\ x(1,t) \\ \vdots \\ x(K-2,t) \\ x(K-1,t) \end{bmatrix} \qquad \bar{y}(t) = \begin{bmatrix} y(0,t) \\ y(1,t) \\ \vdots \\ y(K-2,t) \\ y(K-1,t) \end{bmatrix}$$

Say square



K

$$D_2 = \begin{array}{c} \\ K \end{array} \begin{bmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & & & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{bmatrix}$$

K

$$D_4 = \begin{array}{c} \\ K \end{array} \begin{bmatrix} 6 & -4 & 1 & \cdots & 1 & -4 \\ & & & & & 1 \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & & & & & \\ -4 & 1 & & 1 & -4 & 6 \end{bmatrix}$$

put weight on external force

$$D = \alpha D_2 - \beta D_4$$

set time steps 1

$$\bar{x}(t) = [I - \cancel{\Delta t}\, D]^{-1} \left[\bar{x}(t-1) + \overset{\gamma}{\cancel{\Delta t}}\, \bar{F}_x(\bar{x}(t-1), \bar{y}(t-1))\right]$$

$$\bar{y}(t) = [I - \cancel{\Delta t}\, D]^{-1} \left[\bar{y}(t-1) + \overset{\gamma}{\cancel{\Delta t}}\, \bar{F}_y(\bar{x}(t-1), \bar{y}(t-1))\right]$$

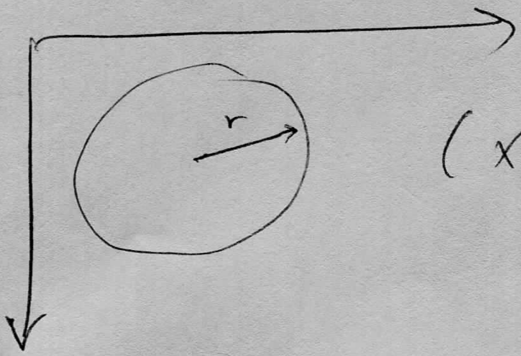
where $\bar{F}(x,y) = \nabla\left[\,\|\nabla f(x,y)\|^2\,\right]$

squared magnitude of gradient

11\6

Level Sets

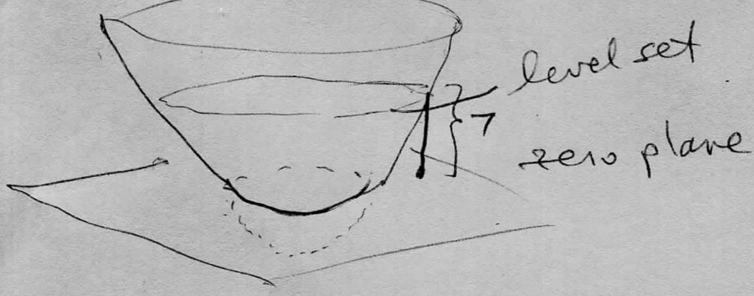
Suppose a circular region:



$$(x-x_0)^2+(y-y_0)^2=r^2$$

Define function of 2 variables

$$\phi(x,y) = (x-x_0)^2+(y-y_0)^2-r^2$$



e.g., suppose $x_0 = y_0 = 0$

& $r=1$

& $x=2$ & $y=2$

then

$\phi(x,y) = 7$

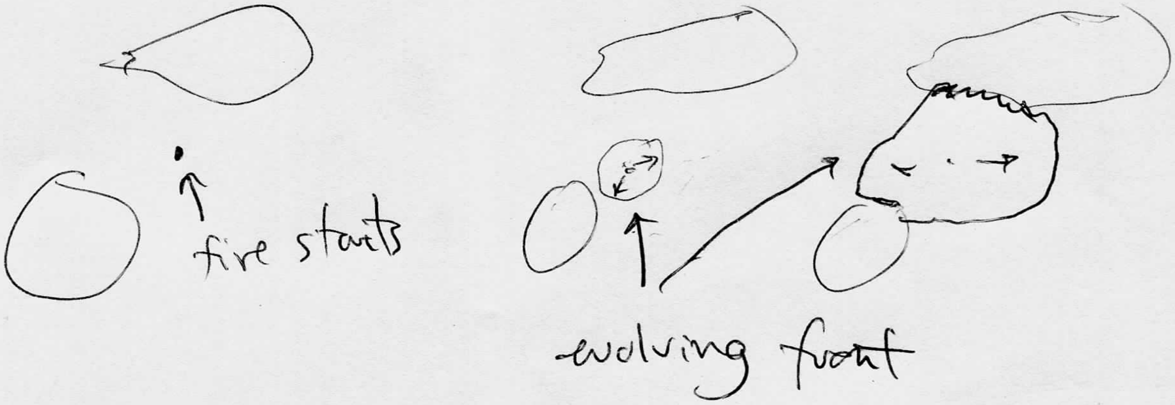
Level set curve $C \equiv \{(x,y) \mid \phi(x,y)=c\}$

say: $\phi(x,y) = \begin{cases} >0 & \text{for } (x,y) \in \Omega^+ \\ =0 & \in \Omega^0 \\ <0 & \in \Omega^- \end{cases}$

$f(x,y) = \Omega^- \cup \Omega^0 \cup \Omega^+$

Like skeletons (!) level sets can be viewed as where a fire quenches while burning through an image (p. 905)

→ at given level set



Consider equation with time:

$$\phi(x(t), y(t), t) = 0$$

let $\bar{z}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$\Rightarrow \phi(\bar{z}(t), t) = 0$$

$$\frac{\partial \phi(\bar{z}(t), t)}{\partial t} = 0$$

$$\frac{\partial \phi}{\partial \bar{z}(t)} \cdot \frac{\partial \bar{z}(t)}{\partial t} + \frac{\partial \phi \, \partial t}{\partial t \, \partial t} = 0$$

$$\frac{\partial \phi}{\partial \bar{z}(t)} \cdot \frac{\partial \bar{z}(t)}{\partial t} + \frac{\partial \phi}{\partial t} = 0$$

$$\underbrace{\frac{\partial \phi}{\partial \bar{z}} = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix}} \qquad \underbrace{\frac{\partial \bar{z}}{\partial t} = \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix}}_{\text{velocity}}$$

$\equiv \nabla \phi$

gradient

gives us a speed function

F acting normal to curve

$$\frac{\partial \bar{z}(t)}{\partial t} = F \bar{n} = F \frac{\nabla \phi}{\|\nabla \phi\|}$$

so, $$\frac{\partial \bar{z}(t)}{\partial t} = F \frac{\nabla \phi}{\|\nabla \phi\|}$$

$$\Rightarrow \quad F \nabla \phi \cdot \frac{\nabla \phi}{\|\nabla \phi\|} + \frac{\partial \phi}{\partial t} = 0$$

but $\nabla \phi \cdot \nabla \phi = \|\nabla \phi\|^2$, so

$$\left.\frac{\partial \phi}{\partial t} + F \|\nabla \phi\| = 0 \quad \right\} \text{level set equation}$$

$$\frac{\partial \phi}{\partial t} = -F \|\nabla \phi\|$$

11\10

Example

level set rep of circle of radius r at origin

$$\phi(x,y) = \sqrt{x^2+y^2} - r$$

CS4640_LS_example_circle

$$\nabla\phi = \begin{bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sqrt{x^2+y^2} - r}{\partial x} \\ \frac{\partial \sqrt{x^2+y^2} - r}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

$$\frac{\partial (x^2+y^2)^{1/2} - r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

So, $$\|\nabla\phi\| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1$$

Level set eqn: $\frac{\partial\phi}{\partial t} = -1$

Solution: $\phi(x,y,t) = \sqrt{x^2+y^2} - r - t$ ← what's the $\frac{\partial\phi}{\partial t}$? $\equiv -1$

Solution of Level Set Equation

$$\frac{\partial \phi(x,y,t)}{\partial t} \doteq \frac{\phi(x,y,t+\Delta t) - \phi(x,y,t)}{\Delta t}$$

Let $\phi^n = \phi(x,y,n\Delta t)$

then $$\frac{\partial \phi^n}{\partial t} = \frac{\phi^{n+1} - \phi^n}{\Delta t}$$

$$\rightarrow \quad \phi^{n+1} = \phi^n + \Delta t \frac{\partial \phi^n}{\partial t}$$

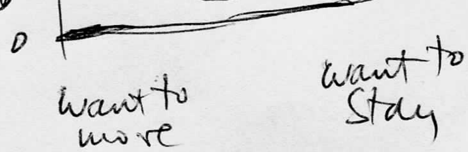
$\phi^0 = \phi(x,y,0)$ [which is? Try example]

$$\phi^{n+1} = \phi^n - \Delta t \left\{ \underbrace{F(x,y)}_{\text{use function of image}} \| \nabla \phi^n \| \right\}$$

converges if $\phi^{n+1} \approx \phi^n$

want to stay at edges

e.g., ~~mag~~



$$F(x,y) = \exp(-\text{mag}(x,y))$$

if 0 → F will be 1

if large → F will be 0

11/12

Solving the Level Set Eqn

Need to consider direction of propagation.

Done as follows:

$$D_x^+ = \phi(x+1,y) - \phi(x,y)$$
$$D_x^- = \phi(x,y) - \phi(x-1,y)$$
$$D_y^+ = \phi(x,y+1) - \phi(x,y)$$
$$D_y^- = \phi(x,y) - \phi(x,y-1)$$

Then

$$\|\nabla\phi\|^+ = \Big([\max(D_x^-,0)]^2 + [\min(D_x^+,0)]^2 + [\max(D_y^-,0)]^2 + [\min(D_y^+,0)]^2 \Big)^{1/2}$$

$$\|\nabla\phi\|^- = \Big([\max(D_x^+,0)]^2 + [\min(D_x^-,0)]^2 + [\max(D_y^+,0)]^2 + [\min(D_y^-,0)]^2 \Big)^{1/2}$$

And:

$$\phi^{n+1} = \phi^n - \Delta t \left\{ \max(F,0)\|\nabla\phi^n\|^+ + \min(F,0)\|\nabla\phi^n\|^- \right\}$$

Make sure $\Delta t < \frac{1}{\max(F)}$