Spatial Filtering

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[Slides borrowed from Ross Whitaker]

Overview

- Correlation and convolution
- Linear filtering
  - Smoothing, kernels, models
  - Detection
  - Derivatives
- Nonlinear filtering
  - Median filtering
  - Bilateral filtering
  - Neighborhood statistics and nonlocal filtering

Cross Correlation

- Operation on image neighborhood and small …
  - "mask", "filter", "stencil", "kernel"
- Linear operations within a moving window

Input image

100	130	104	99	…
87	95	103	150	…
50	36	150	104	…
20	47	205	77	…
.	.	.	.	
.	.	.	.	

Filter

0.0	0.1	0.0
0.1	0.6	0.1
0.0	0.1	0.0

Output image

.	.	.	.
.	.	.	.
.	55.8	134.8	.
.	.	.	.

```
  0.0*87 + 0.1*95 + 0.0*103
+ 0.1*50 + 0.6*36 + 0.1*150
+ 0.0*20 + 0.1*47 + 0.0*205 = 55.8 134.8
```

Cross Correlation

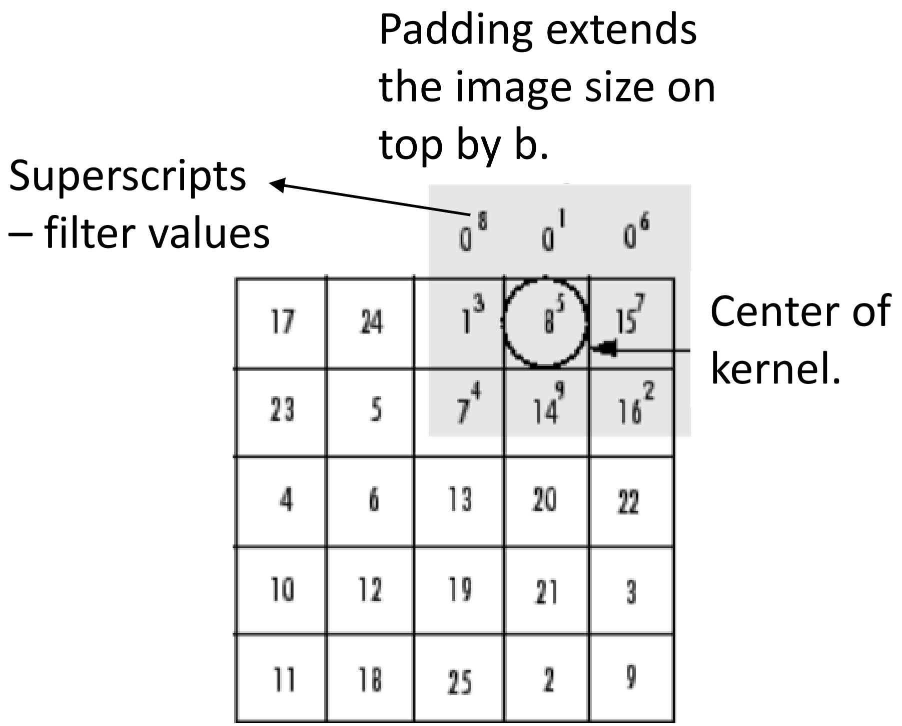
- 1D $$g(x) = \sum_{s=-a}^{a} w(s) f(x+s)$$

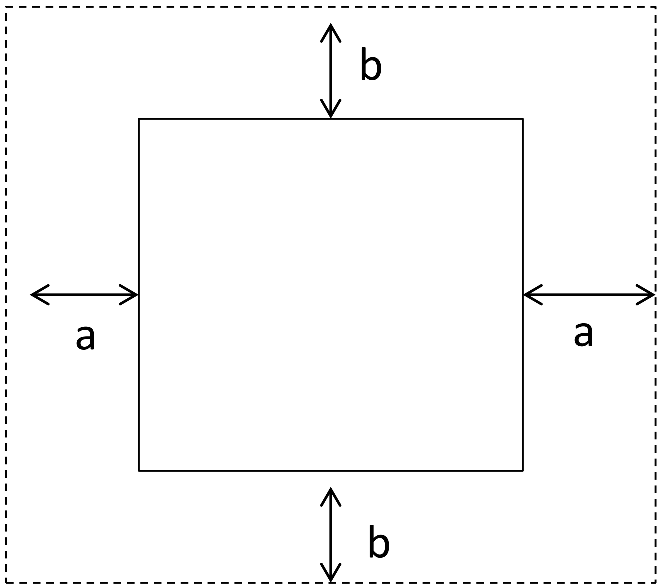
- 2D $$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t)$$

$$w(s,t) = \begin{matrix} w(-a,-b) & \cdots & & \cdots & w(a,-b) \\ \vdots & & & & \vdots \\ & \cdots & w(0,0) & \cdots & \\ \vdots & & & & \vdots \\ w(-a,b) & \cdots & & \cdots & w(a,b) \end{matrix}$$

Correlation: Technical Details

– Boundary conditions – Pad image with amount (a,b) for a filter of size (2a+1,2b+1).



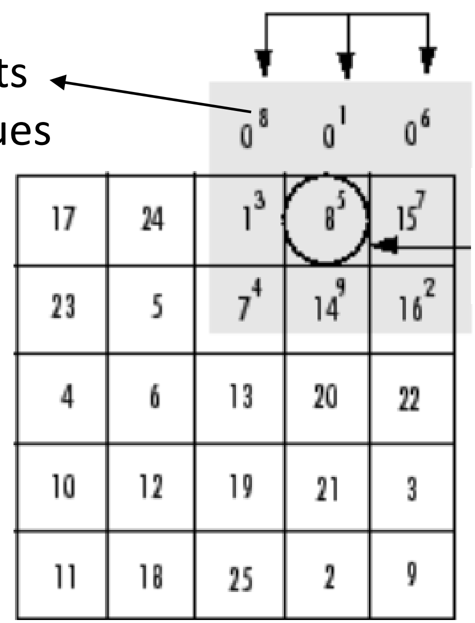


Correlation: Technical Details

- Boundary conditions – zero padding, replication of boundary pixels

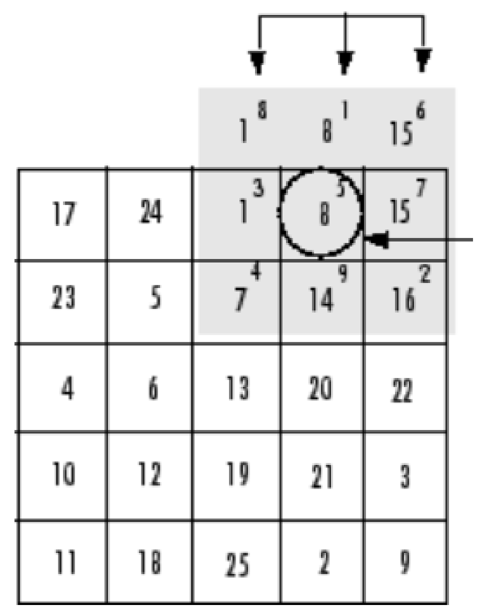
Outside pixels are assumed to be 0's.

Superscripts – filter values



Center of kernel.

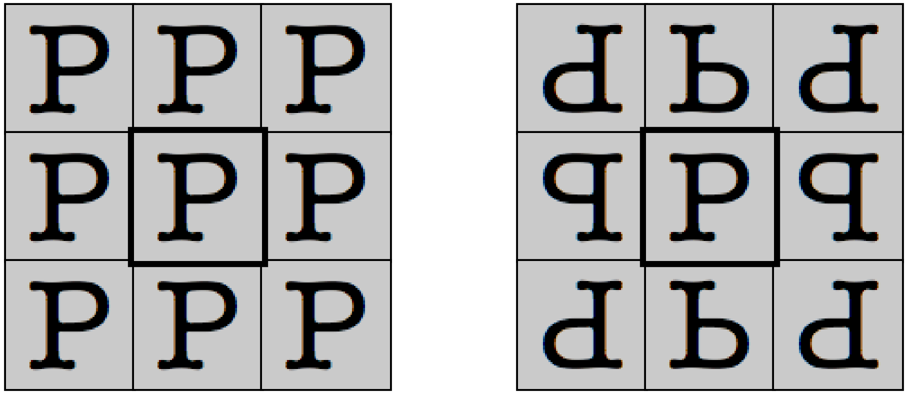
The pixel values are replicated from boundary pixels.



Center of kernel.

Correlation: Technical Details

- Boundary conditions
  - Pad image with amount (a,b)
    - Constant value or repeat edge values
  - Cyclical boundary conditions
    - Wrap or mirroring



Correlation: Technical Details

- Boundaries
  - Can also modify kernel – no long correlation
- For analysis
  - Image domains infinite
  - Data compact (goes to zero far away from origin)

$$g(x,y) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} w(s,t) f(x+s, y+t)$$

Correlation: Properties

- Shift invariant – discrete equivalent of time invariant system - If $x(t)$ gives $y(t)$, then $x(t+t_0)$ gives y$(t+t_0)$.

$$g = w \circ f$$ New notation for correlation

$$w \circ f(x-x_0, y-y_0) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} w(s,t) f(x-x_0+s, y-y_0+t) = g(x-x_0, y-y_0)$$

- Linear $w \circ (\alpha e + \beta f) = \alpha w \circ e + \beta w \circ f$

$$C_{wf} = w \circ f$$ Compact notation

Cross Correlation Continuous Case

- f, w must be "integrable"
  - Must die off fast enough so that integral is finite

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(s,t)f(x+s,y+t)dsdt$$

- Same properties as discrete case
  - Linear
  - Shift invariant

Filters: Considerations

- Normalize
  - Sums to one
  - Sums to zero (some cases, later)
- Symmetry
  - Left, right, up, down
  - Rotational
- Special case: auto correlation

$$C_{ff} = f \circ f$$

Examples 1

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$1/9 * \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Examples 2

$$\frac{1}{9} * \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

$$\frac{1}{25} * \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

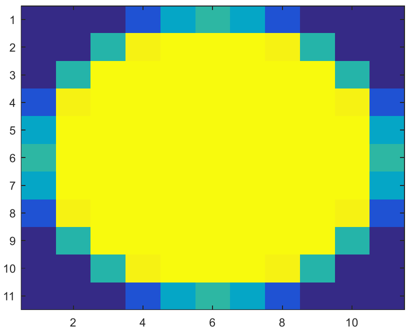
Smoothing and Noise

Noisy image

5x5 box filter

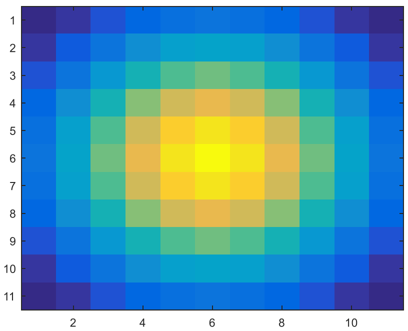
Other Filters

- Disk
  - Circularly symmetric, jagged in discrete case



- Gaussians
  - Circularly symmetric, smooth for large enough std-dev.
  - Must normalize in order to sum to one

$$\frac{1}{2\pi\sigma}\exp -\frac{\left(x^2+y^2\right)}{2\sigma^2}$$



- Derivatives – discrete/finite differences
  - Operators

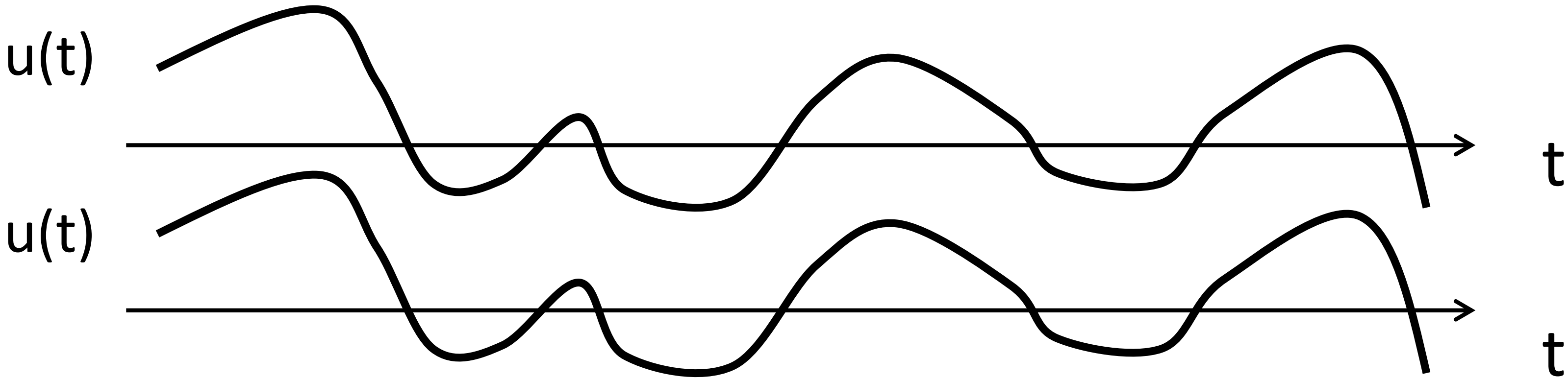
Pattern Matching/Detection

- The optimal (highest) response from a filter is the autocorrelation evaluated at position zero

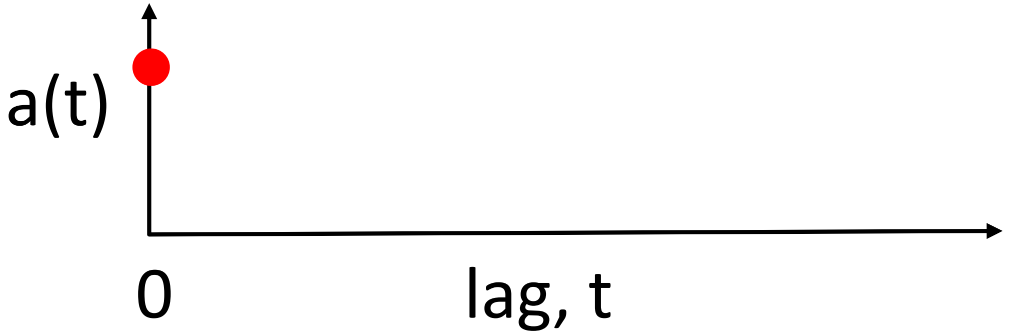
$$\max_{\bar{x}} C_{ff}(\bar{x}) = C_{ff}(0) = \int f(\bar{s}) f(\bar{s}) d\bar{s}$$

- A filter responds best when it matches a pattern that looks like itself
- Strategy
  - Detect objects in images by correlation with "matched" filter

Measure of correlation in time series at different lags

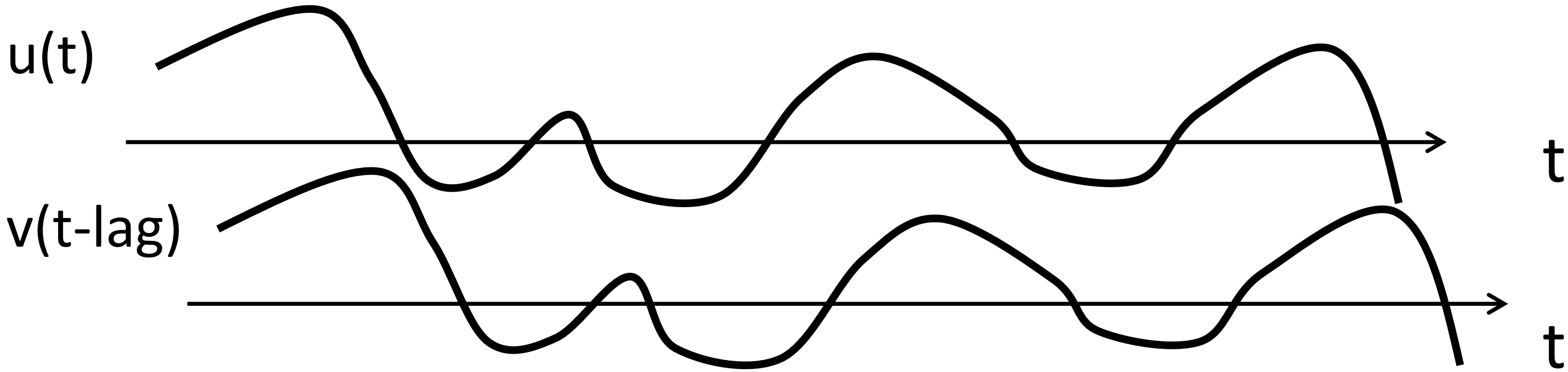


No lag, multiply and sum area

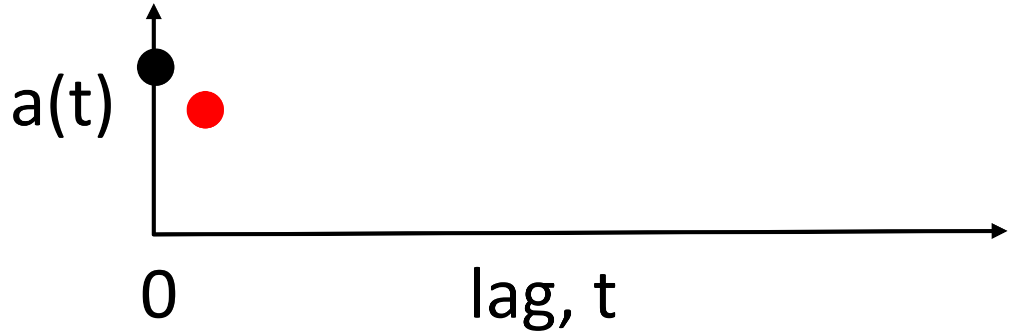


Source: http://www.ldeo.columbia.edu/users/menke/edawm/index.htm

Measure of correlation in time series at different lags

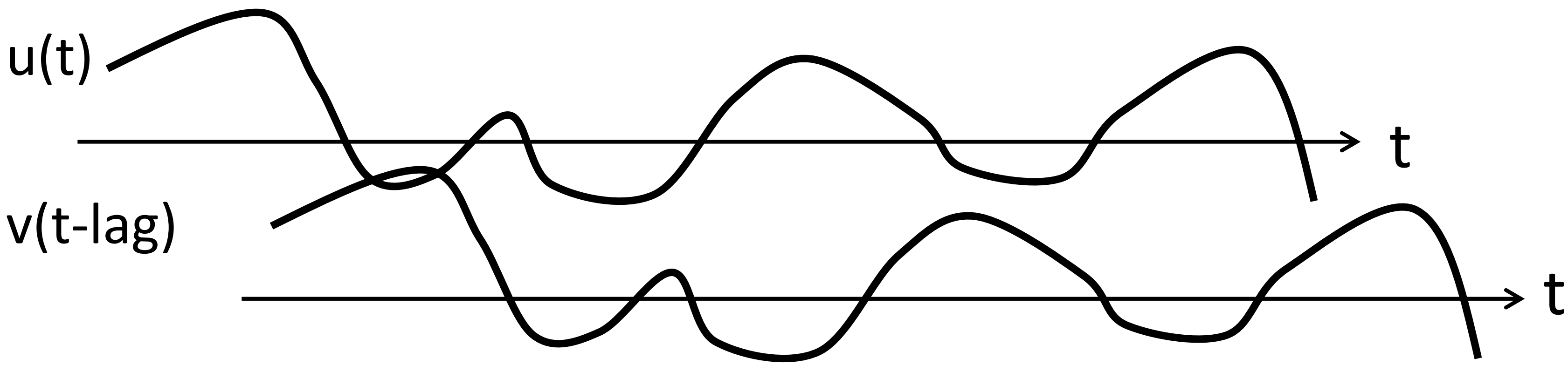


Small lag, multiply and sum area

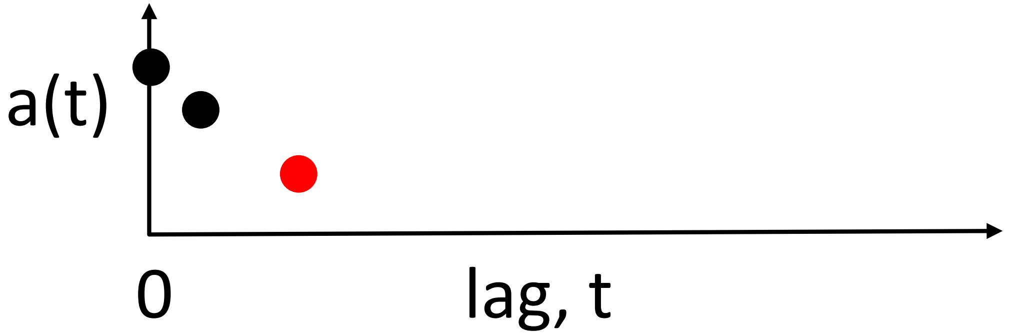


Source: http://www.ldeo.columbia.edu/users/menke/edawm/index.htm

Measure of correlation in time series at different lags

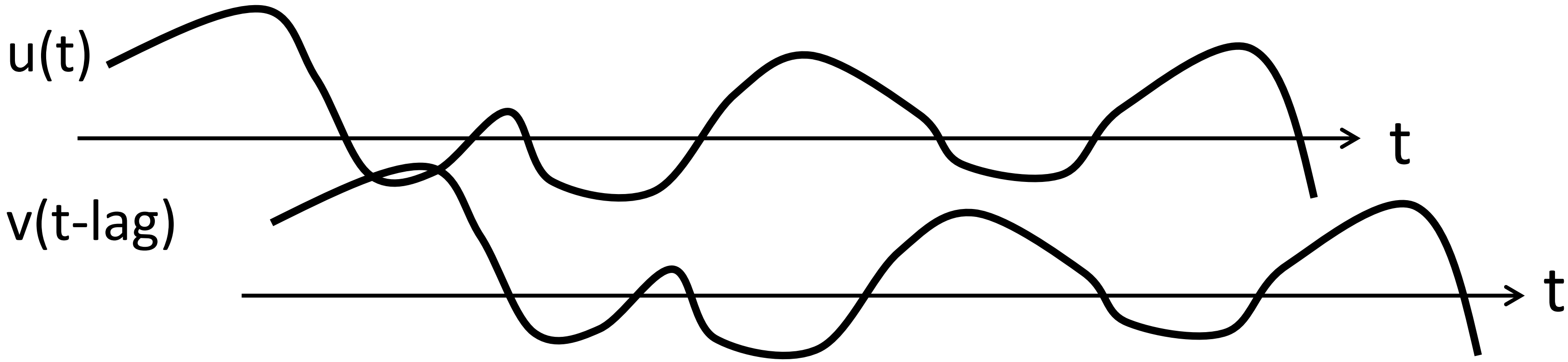


Large lag, multiply and sum area

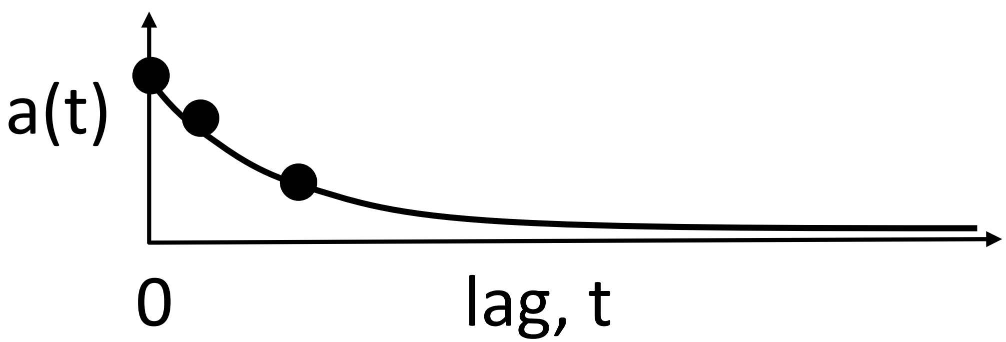


Source: http://www.ldeo.columbia.edu/users/menke/edawm/index.htm

Measure of correlation in time series at different lags



lag, multiply and sum area



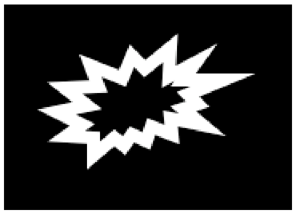
Source: http://www.ldeo.columbia.edu/users/menke/edawm/index.htm

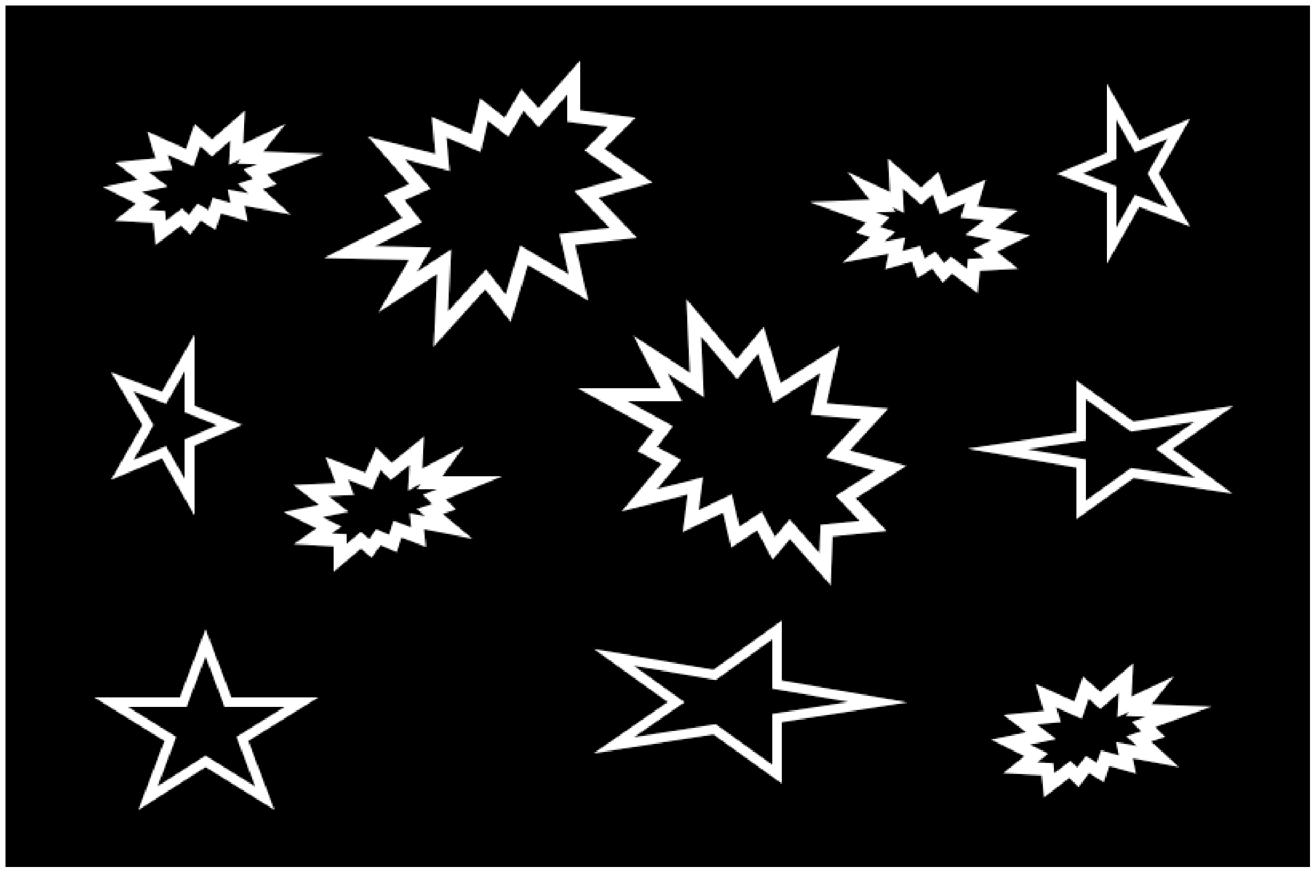
Reasoning

- Dot product of two normalized vectors a and b is maximum when both the vectors are equal.

$$a.b = |a||b|\cos(\theta)$$

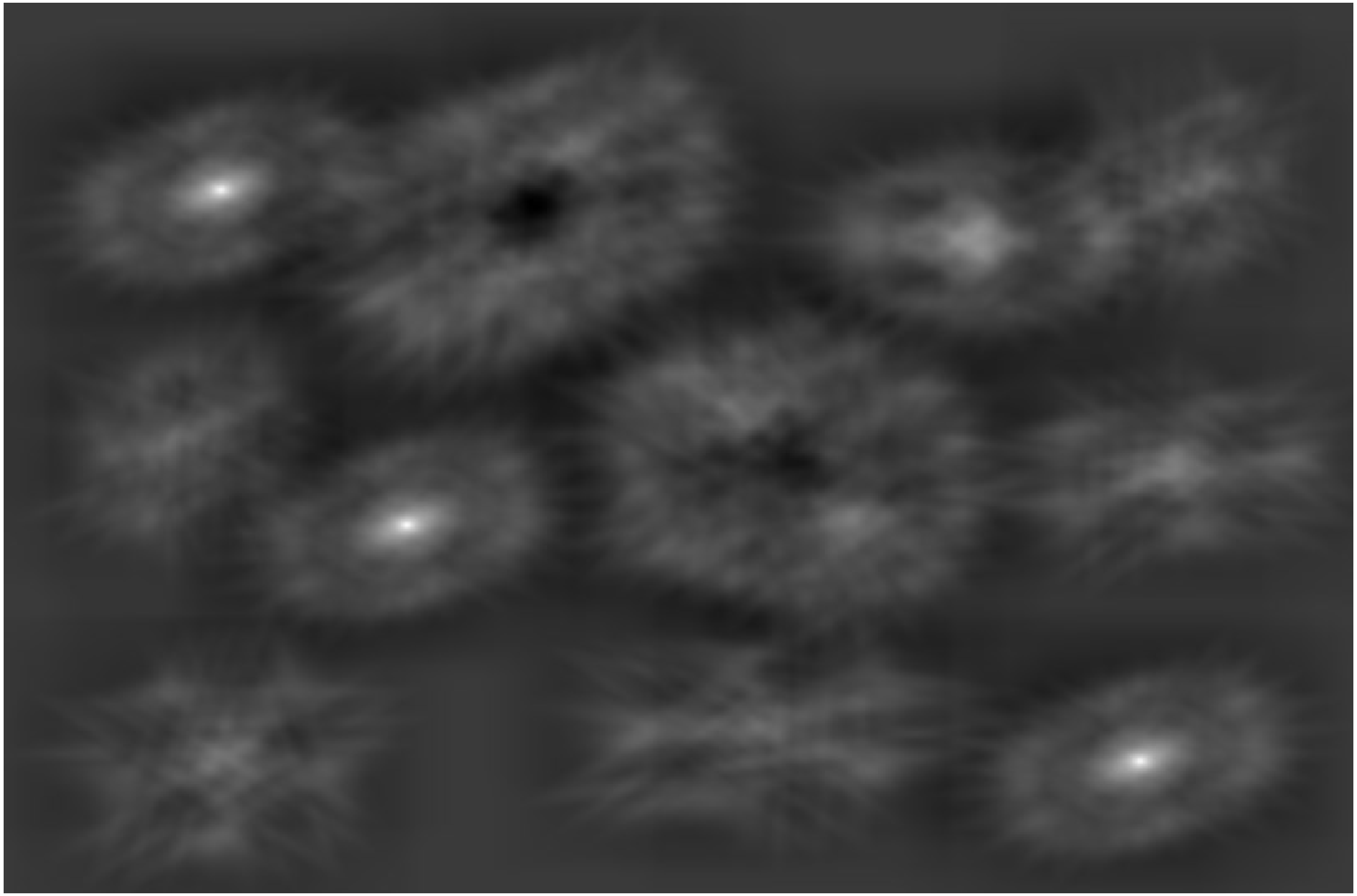
Match Filter Example





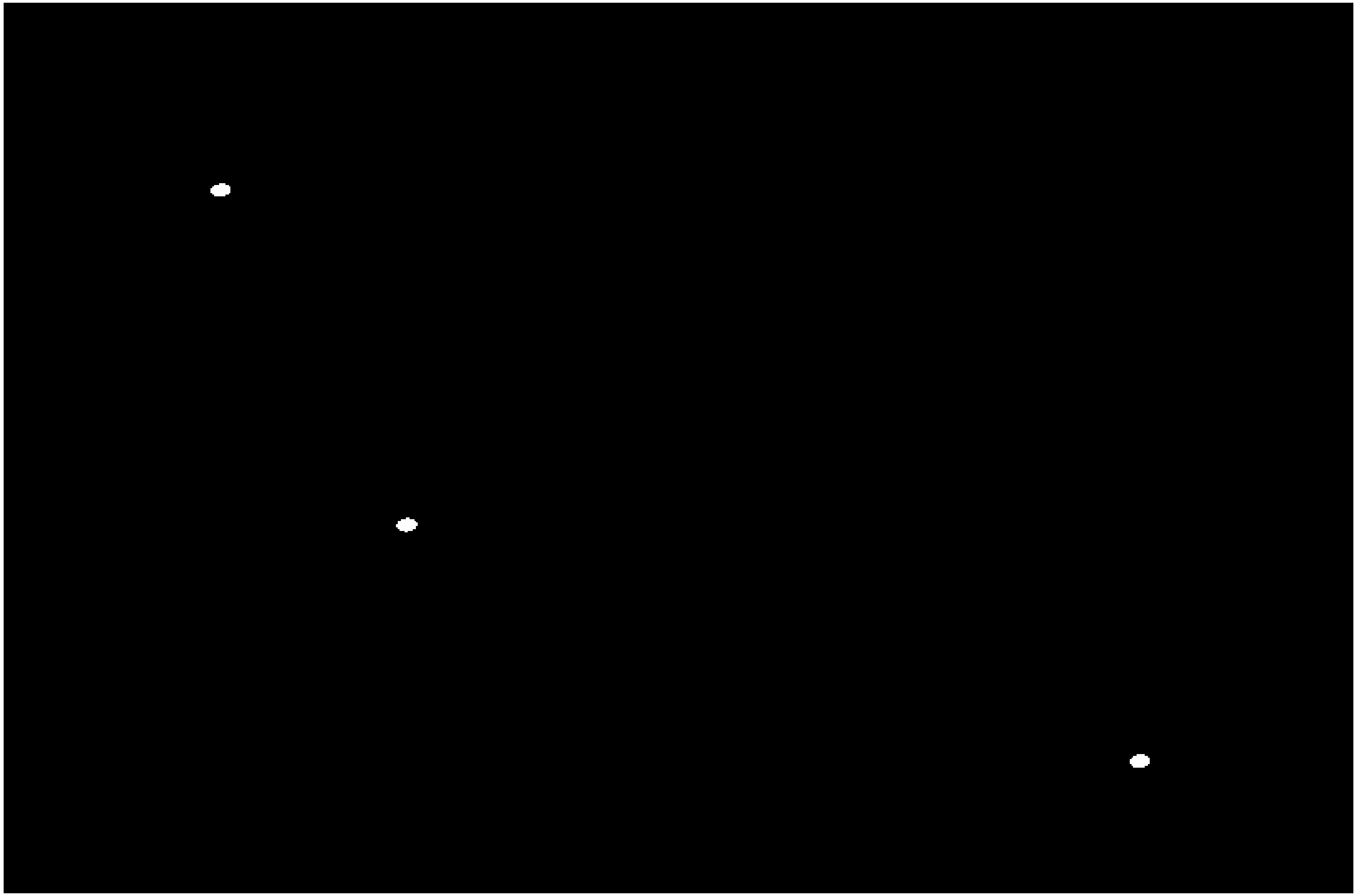
Trick: make sure kernel sums to zero

Match Filter Example



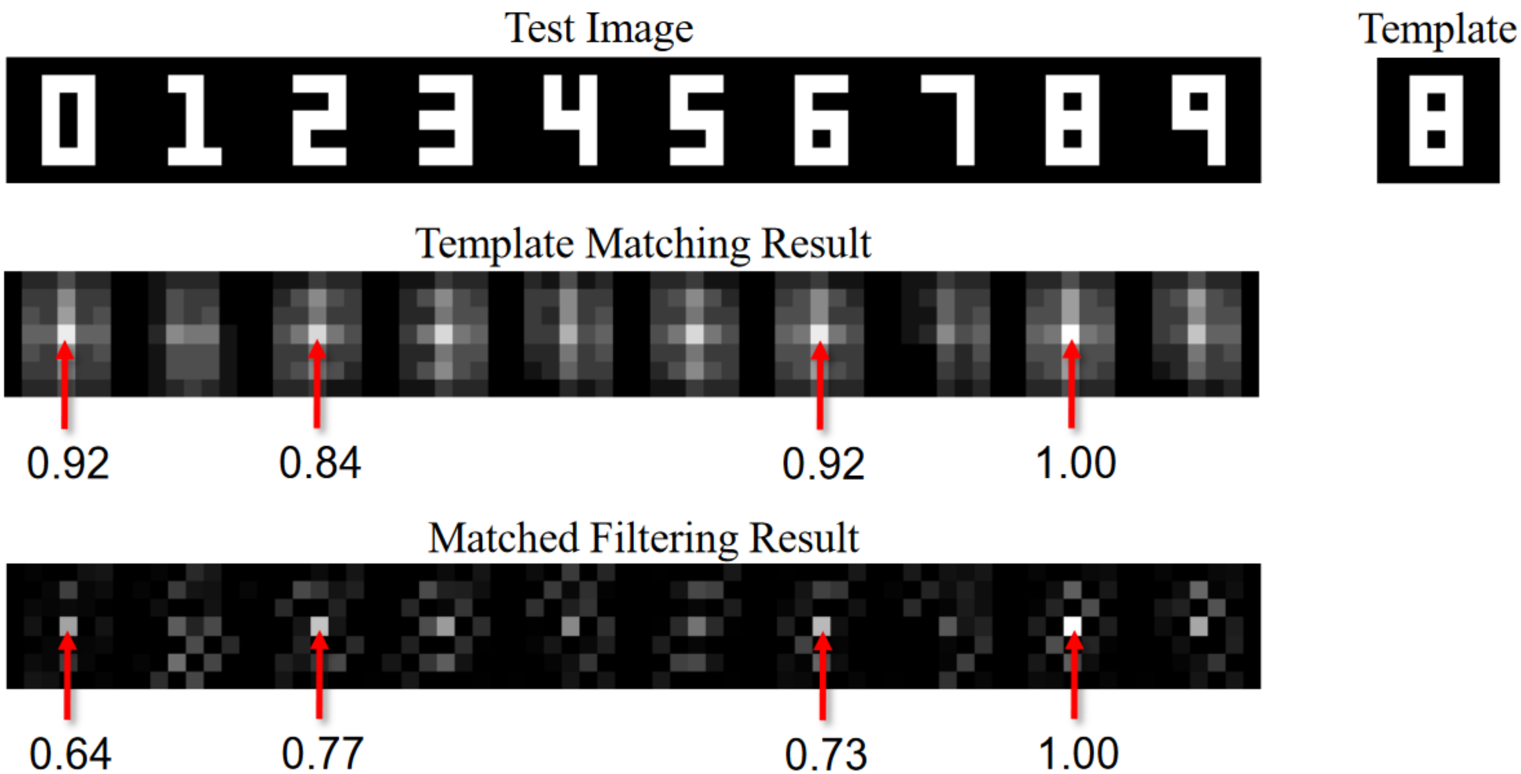
[Responses from match filtering]

Match Filter Example



[Peaks from match filtering]

Match Filter Example



[Source: Bernd Girod]

Derivatives: Finite Differences

$$\frac{\partial f}{\partial x} \approx \frac{f(x+1,y) - f(x-1,y)}{2}$$

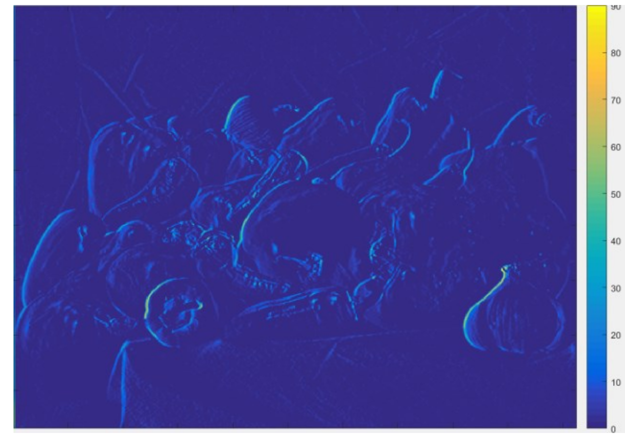
$$\frac{\partial f}{\partial x} \approx \frac{f(x,y+1) - f(x,y-1)}{2}$$

$$\frac{\partial f}{\partial x} \approx w_{dx} \circ f \qquad w_{dx} = \begin{array}{|c|c|c|}\hline -\frac{1}{2} & 0 & \frac{1}{2} \\ \hline \end{array}$$

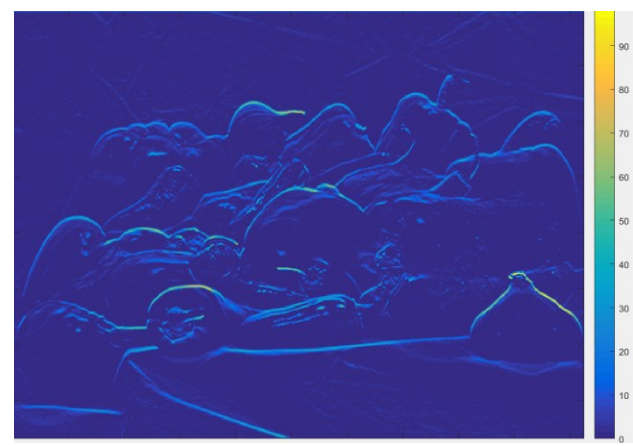
$$\frac{\partial f}{\partial y} \approx w_{dy} \circ f \qquad w_{dy} = \begin{array}{|c|}\hline -\frac{1}{2} \\ \hline 0 \\ \hline \frac{1}{2} \\ \hline \end{array}$$

Derivative Example

$$\begin{matrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$



Other filters

- Prewitt

-1	0	+1
-1	0	+1
-1	0	+1

-1	-1	-1
0	0	0
+1	+1	+1

- Sobel

-1	0	+1
-2	0	+2
-1	0	+1

-1	-2	-1
0	0	0
+1	+2	+1

Convolution

- Discrete

$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

- Continuous

$$g(x,y) = w(x,y) * f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(s,t) f(x-s, y-t) ds dt$$

- Same as cross correlation with kernel transposed around each axis

Convolution = Correlation

- The two operations (correlation and convolution) are the same if the kernel is symmetric about axes

$$g = w \circ f = w^* * f$$

w^* reflection of w

Convolution: Properties

- Shift invariant, linear
- Commutative

$$f * g = g * f$$

- Associative

$$f * (g * h) = (f * g) * h$$

- Others (discussed later):
  - Derivatives, convolution theorem, spectrum…

Associativity

Example: Correlation

[-1 3 -3 1]

[1 -2 1]

[-1 1] [1 -1] [-1 1]

[1 -2 1]

[1 -3 3 -1]

Associativity

Example: Convolution

[1 -3 3 -1]

[-1 2 -1]

[-1 1] [1 -1] [-1 1]

[-1 2 -1]

[1 -3 3 -1]

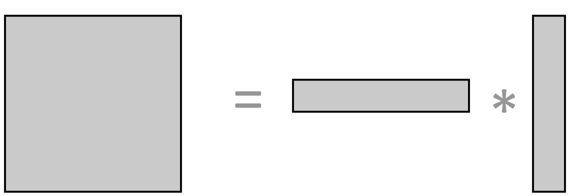
Computing Convolution

- Compute time
  - MxM mask
  - NxN image

  $\Bigg\}$ $O(M^2N^2)$ "for" loops are nested 4 deep

- Special case: separable

$$w = \overbrace{w_x * w_y}^{\text{Two 1D kernels}}$$

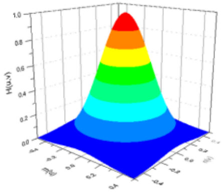


$$w * f = \underbrace{(w_x * w_y) * f}_{O(M^2N^2)} = \underbrace{w_x * (w_y * f)}_{O(MN^2)}$$

Separable Kernels

- Examples
  - Box/rectangle
  - Bilinear interpolation
  - Combinations of partial derivatives
    - d2f/dxdy
  - Gaussian
    - Only filter that is both circularly symmetric and separable

- Counter examples
  - Disk
  - Cone



  - Pyramid

Examples of Separable Kernels

Smoothing Filter

$$\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} * \frac{1}{3}\begin{bmatrix}1 & 1 & 1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

Sobel Filter

$$\mathbf{G_x} = \begin{bmatrix}1 & 0 & -1\\2 & 0 & -2\\1 & 0 & -1\end{bmatrix} * A = \begin{bmatrix}1\\2\\1\end{bmatrix} * \begin{bmatrix}+1 & 0 & -1\end{bmatrix} * A$$

Examples of Separable filter

Prewitt Filter

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} = ?$$

$$\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = ?$$

2nd order derivatives

Second derivatives:

$$\frac{\partial f}{\partial x} = [-1\ 1] * f = f(x) - f(x-1)$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = [-1\ 1] * [-1\ 1] * f = [1\ \ -2\ \ \ 1] * f$$

$$= f(x+1) - 2f(x) + f(x-1)$$

$$\frac{\partial f}{\partial y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} * f = f(y) - f(y-1)$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} * \begin{bmatrix} -1 \\ 1 \end{bmatrix} * f = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} * f$$

$$= f(x+1) - 2f(x) + f(x-1)$$

Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= f(x+1, y) + f(x-1, y) - 2f(x, y)$$

$$+ f(x, y+1) + f(x, y-1) - 2f(x, y)$$

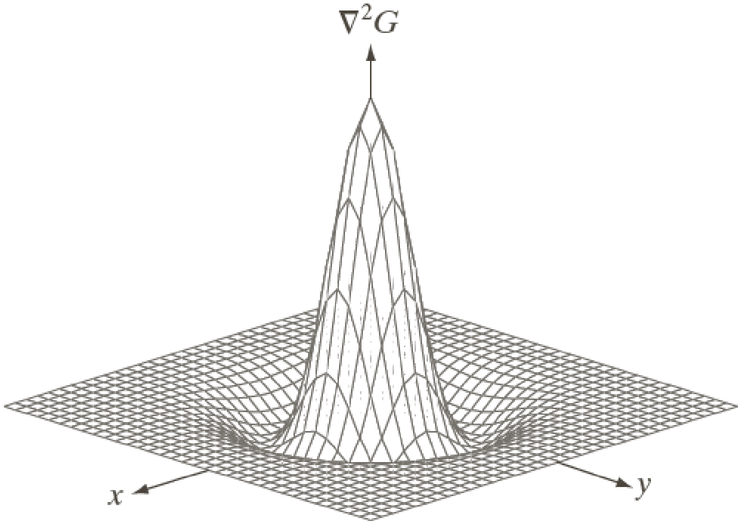
$$= f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1) - 4f(x, y)$$

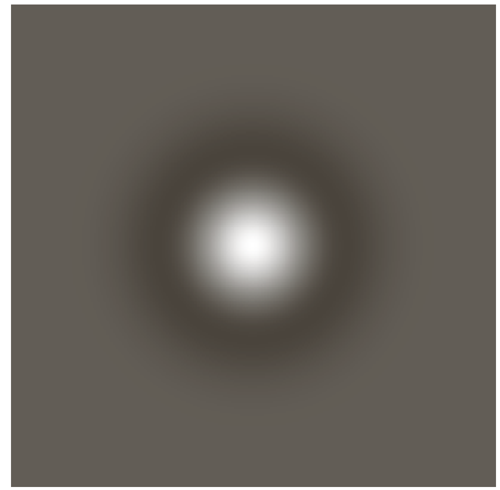
0	**+1**	**0**
+1	**-4**	**+1**
0	**+1**	**0**

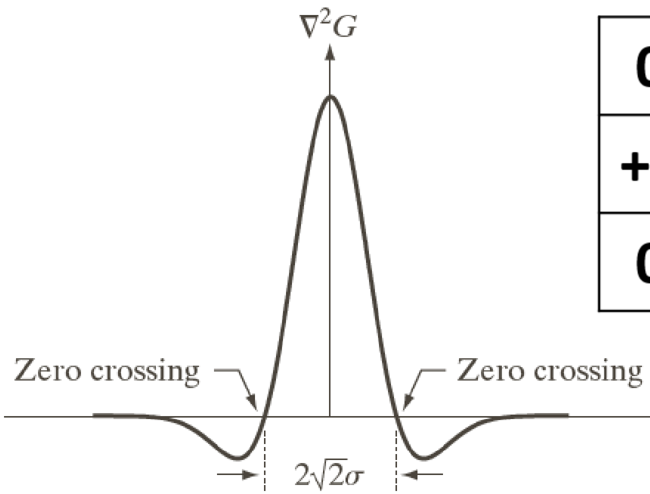
Laplacian of a Gaussian

- We take an image and blur it a little using Gaussian
- function.
- Calculate the second order derivatives or the Laplacian.
- locates edges and corners that are good for detecting keypoints.
- Computation of the second order derivative is also extremely sensitive to noise, and the blurring helps.

Laplacian of Gaussian







0	**+1**	**0**
+1	**-4**	**+1**
0	**+1**	**0**

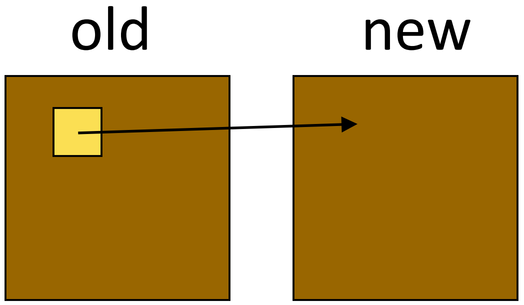
$$* \quad \frac{1}{2\pi\sigma}\exp\left[-\frac{\left(x^2+y^2\right)}{2\sigma^2}\right]$$

Nonlinear Methods For Filtering

- Median filtering
- Bilateral filtering
- Neighborhood statistics and nonlocal filtering

Median Filtering

- For each neighborhood in image
  - Sliding window
  - Usually odd size (symmetric) 5x5, 7x7,…
- Sort the greyscale values
- Set the center pixel to the median
- Important:
  - Separate input and output buffers
  - All statistics on the original image

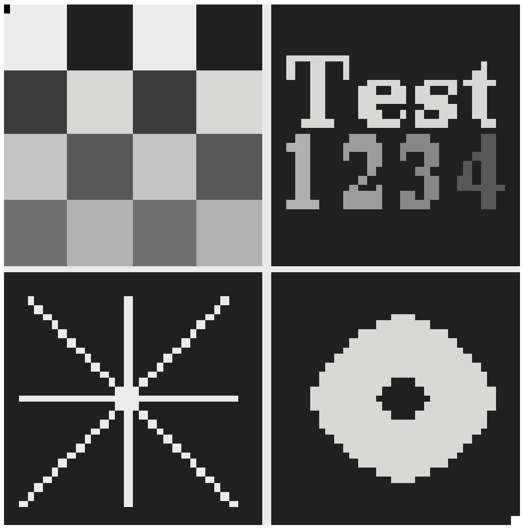


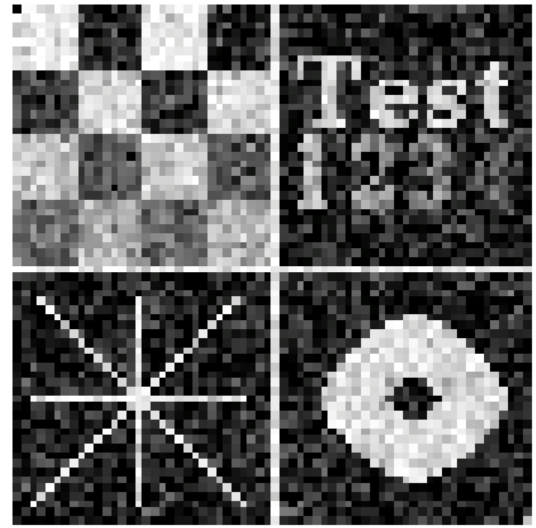
Median Filter

- Issues
  - Boundaries
    - Compute on pixels that fall within window
  - Computational efficiency
    - What is the best algorithm?
- Properties
  - Removes outliers (replacement noise – salt and pepper)
  - Window size controls size of structures
  - Preserves straight edges, but rounds corners and features

Median vs Gaussian

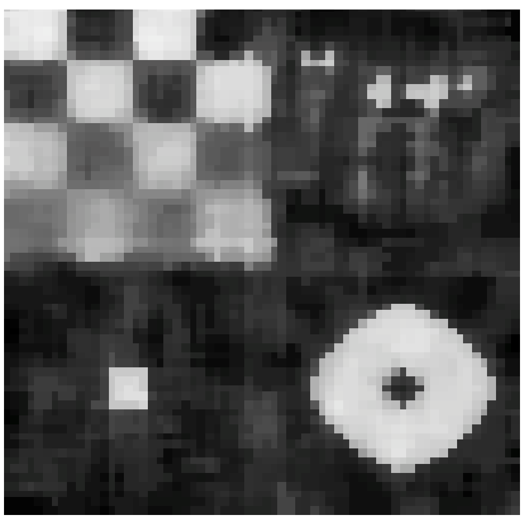
Original

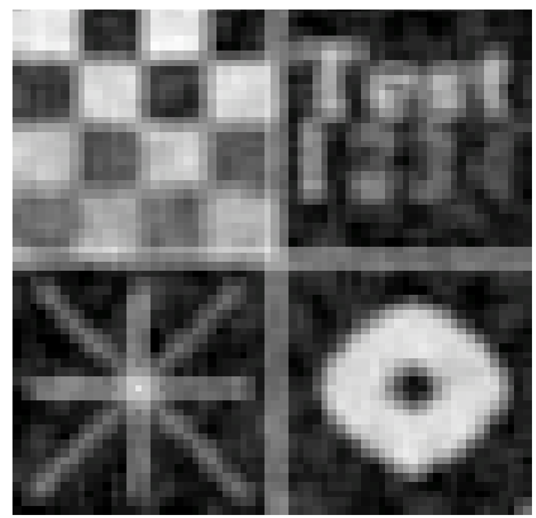




+ Gaussian Noise

3x3 Median

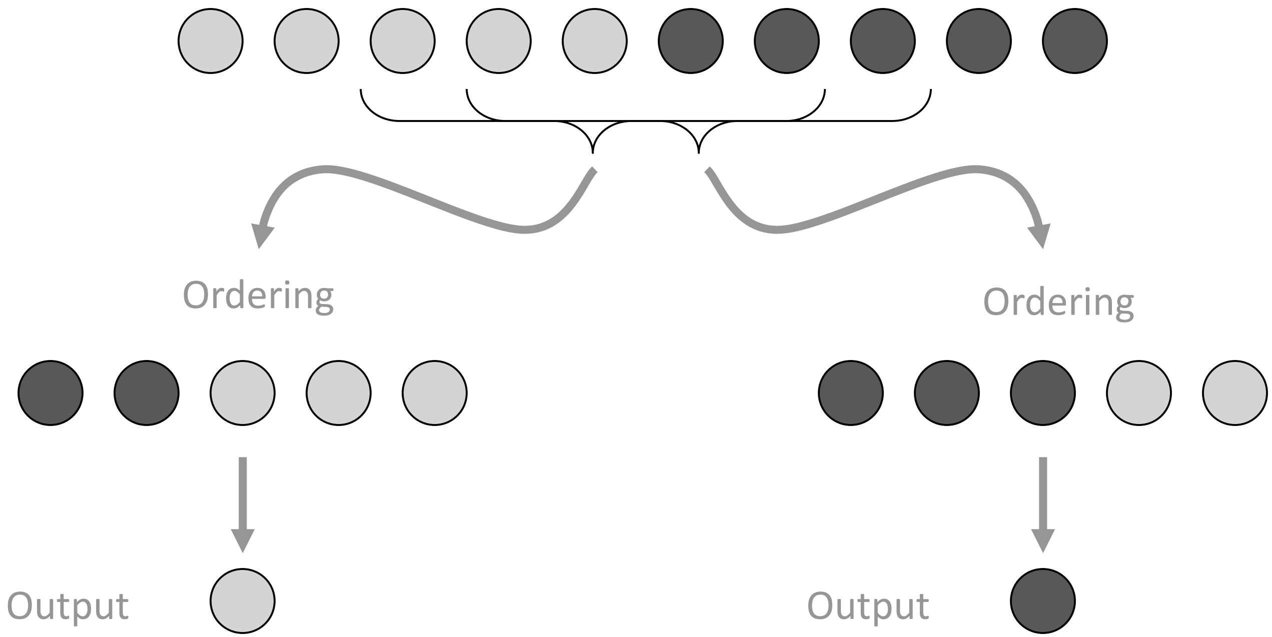




3x3 Box

Median Filtering

- Image model: piecewise constant (flat)



Order Statistics

- Median is special case of order-statistics filters
- Instead of weights based on neighborhoods, weights are based on ordering of data

Neighborhood

$$X_1, X_2, \ldots, X_N$$

Ordering

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$$

Filter

$$F(X_1, X_2, \ldots, X_N) = \alpha_1 X_{(1)} + \alpha_2 X_{(2)} + \ldots + \alpha_N X_{(N)}$$

Neighborhood average (box)

$$\alpha_i = 1/N$$

Median filter

$$\alpha_i = \begin{cases} 1 & i = (N+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Trimmed average (outlier removal)

$$\alpha_i = \begin{cases} 1/M & (N-M+1)/2 \leq i \leq (N+M+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Piecewise Flat Image Models

- Image piecewise flat -> average only within similar regions
- Problem: don’t know region boundaries

Piecewise-Flat Image Models

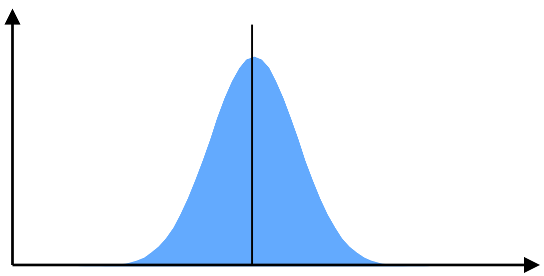
- Assign probabilities to other pixels in the image belonging to the same region
- Two considerations
  - Distance: far away pixels are less likely to be same region
  - Intensity: pixels with different intensities are less likely to be same region

Piecewise-Flat Images and Pixel Averaging

Distance (kernel/pdf)

$$G(\mathbf{x}_i - \mathbf{x}_j)$$

Prob pixel belongs to same region as i

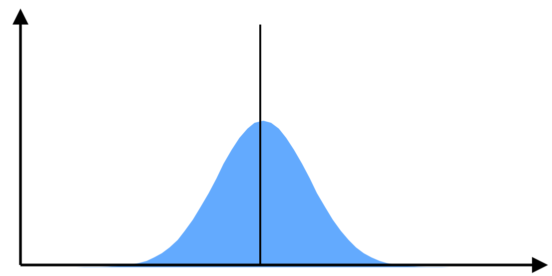


position

Distance (pdf)

$$H(f_i - f_j)$$

Prob pixel belongs to same region as i



intensity

Bilateral Filter

- Neighborhood – sliding window
- Weight contribution of neighbors according to:

$$f_i \leftarrow k_i^{-1} \sum_{j \in N} f_j G(\mathbf{x}_i - \mathbf{x}_j) H(f_i - f_j)$$

$$k_i = \sum_{j \in N} G(\mathbf{x}_i - \mathbf{x}_j) H(f_i - f_j)$$

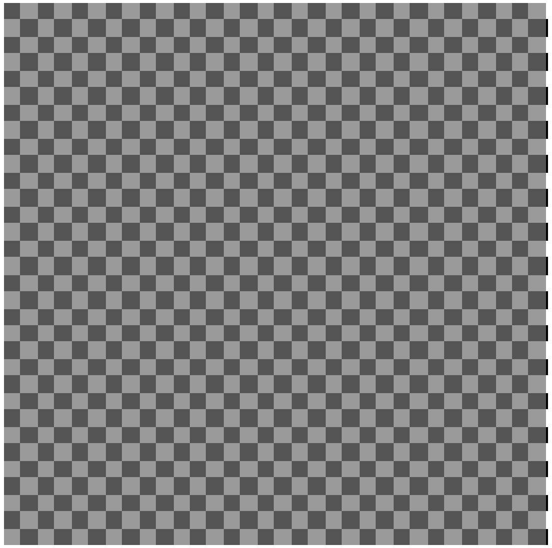
- G is a Gaussian (or lowpass), as is H, N is neighborhood,
  - Often use G(rij) where rij is distance between pixels
  - Update must be normalized for the samples used in this (particular) summation
- Spatial Gaussian with extra weighting for intensity
  - Weighted average in neighborhood with downgrading of intensity outliers

Bilateral Filtering

Gaussian Blurring

Bilateral

Results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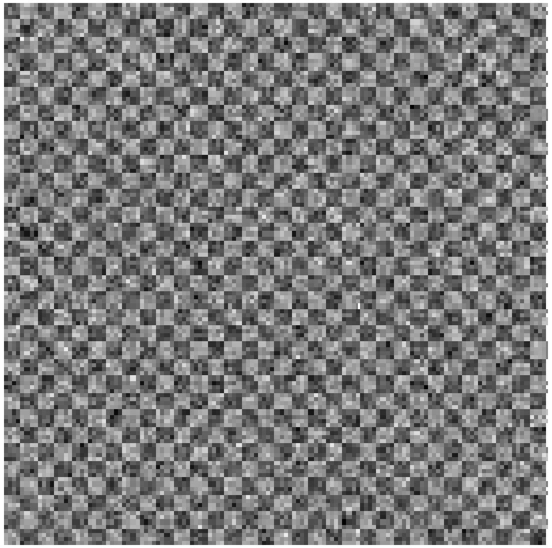
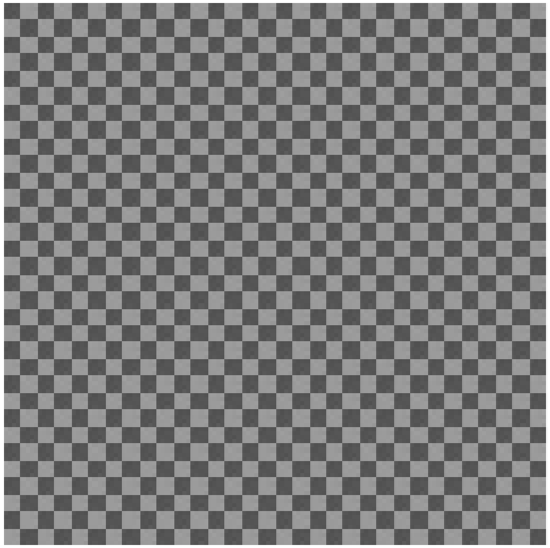
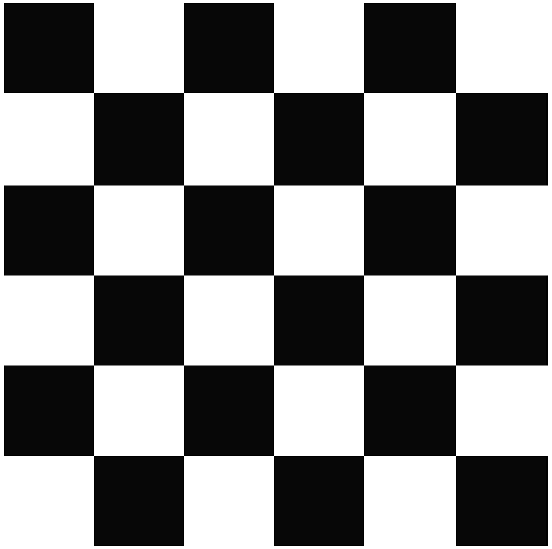

Noisy


Filtered

Checkerboard With Noise



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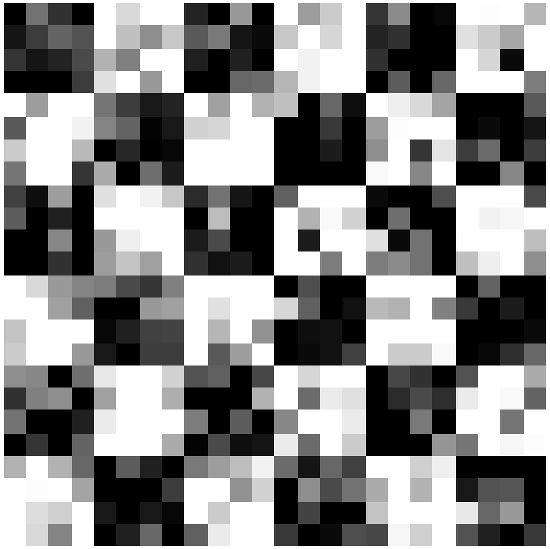

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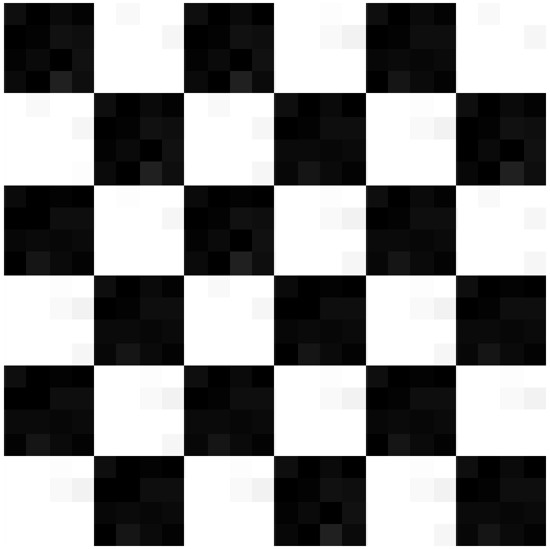

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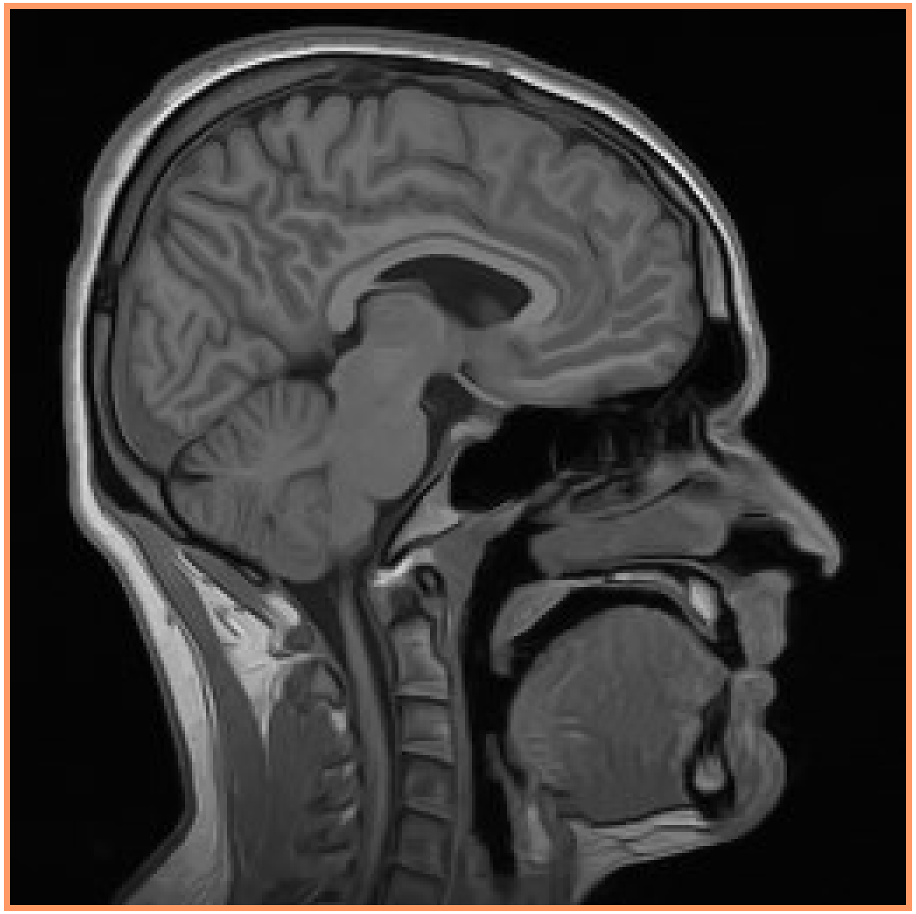
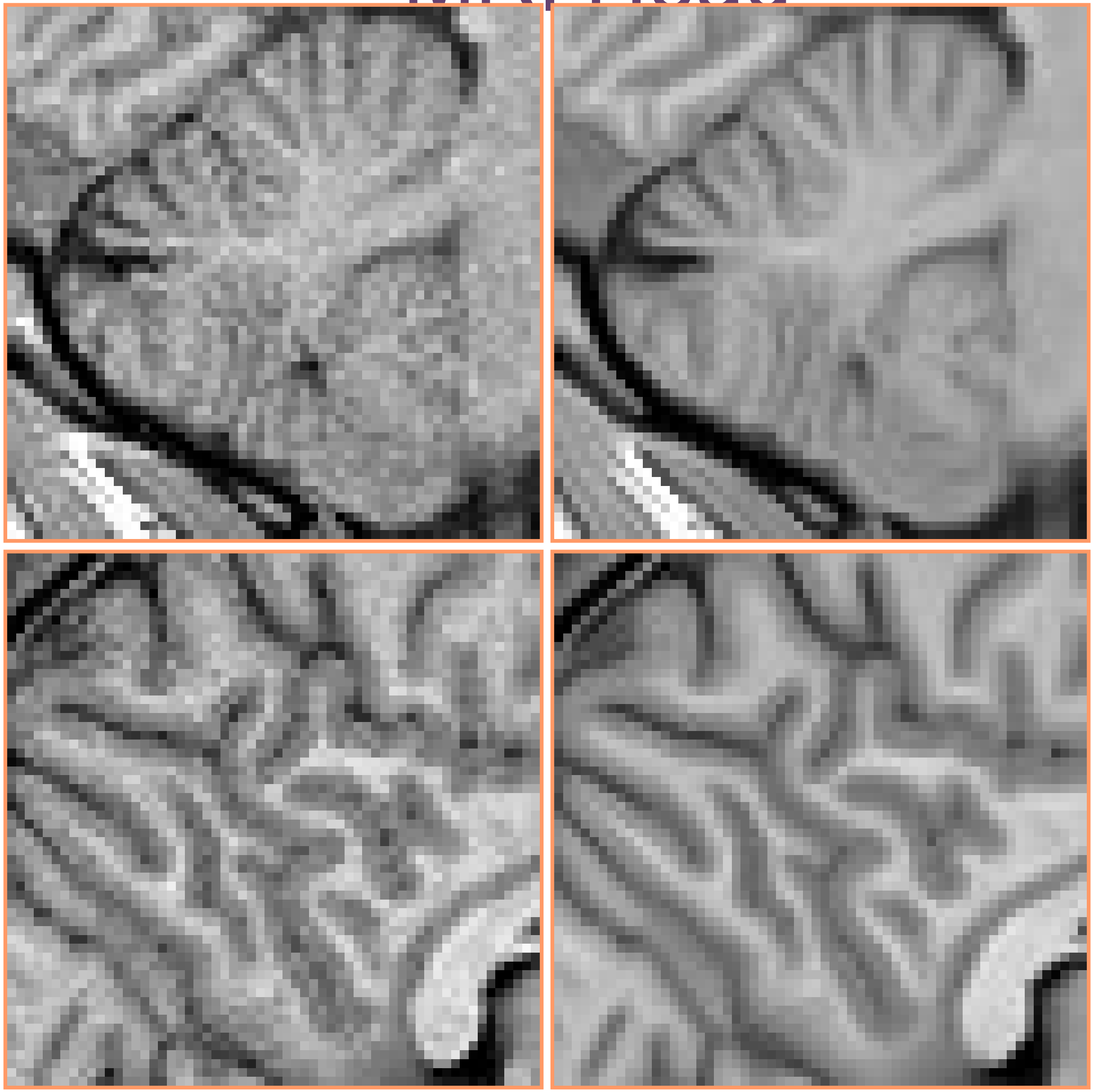
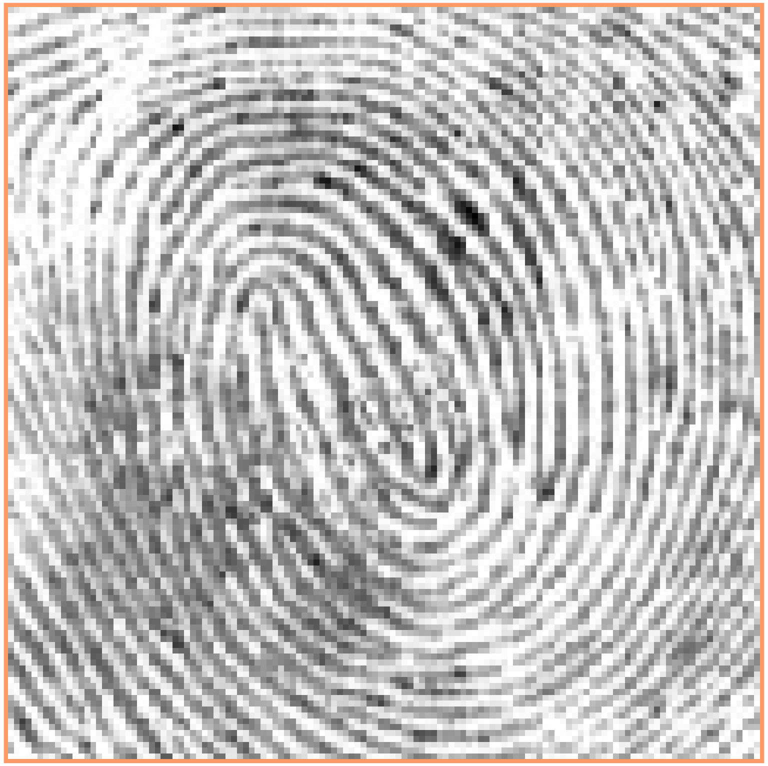
MRI Head



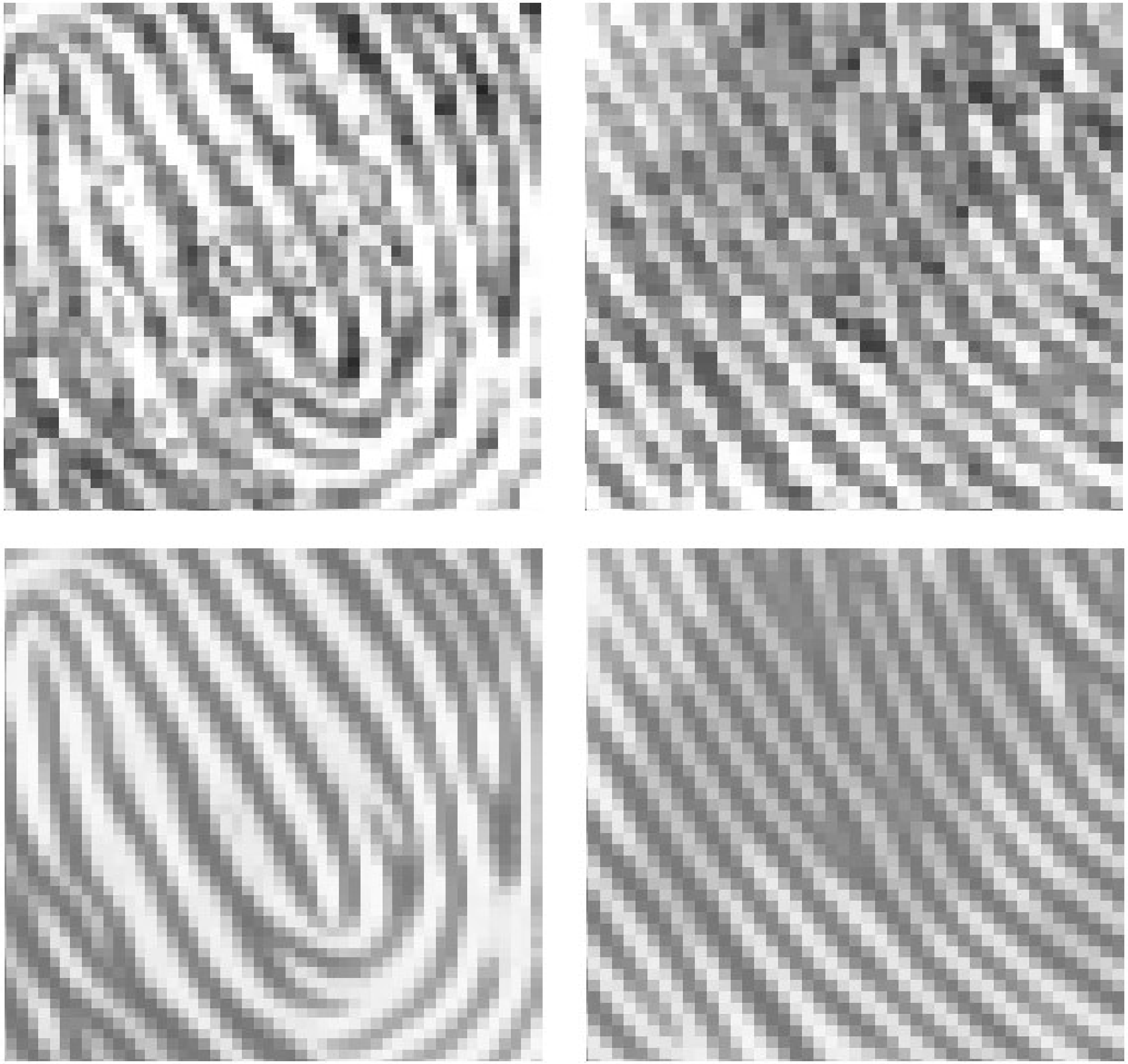
MRI Head



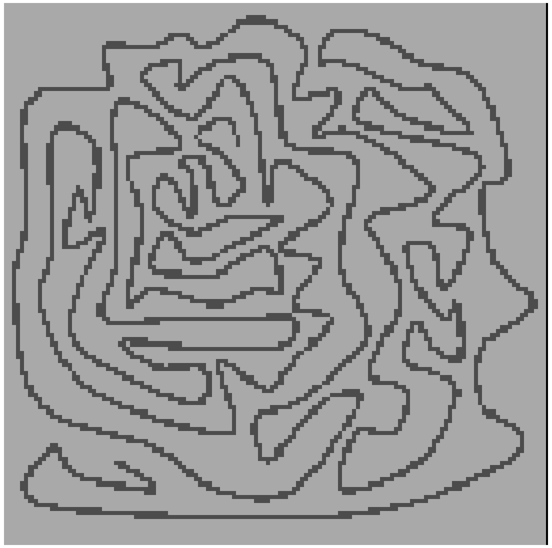
Fingerprint



Fingerprint



Results



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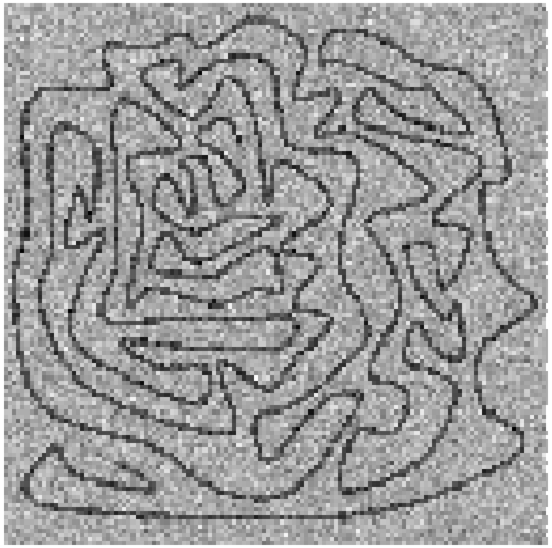

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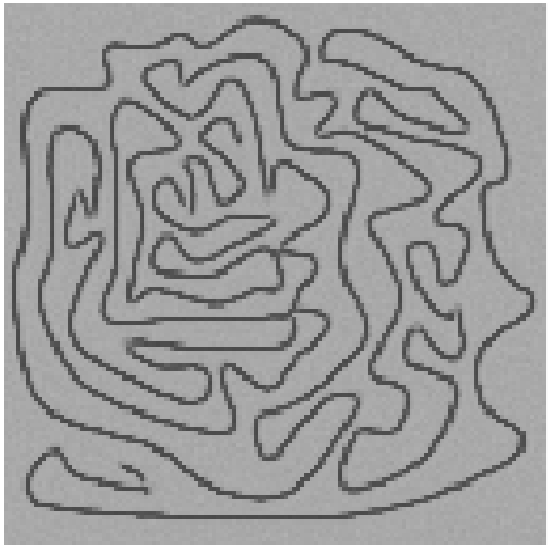

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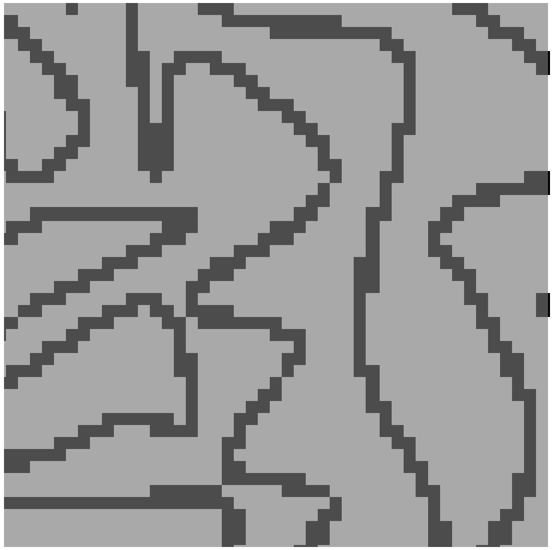
Results



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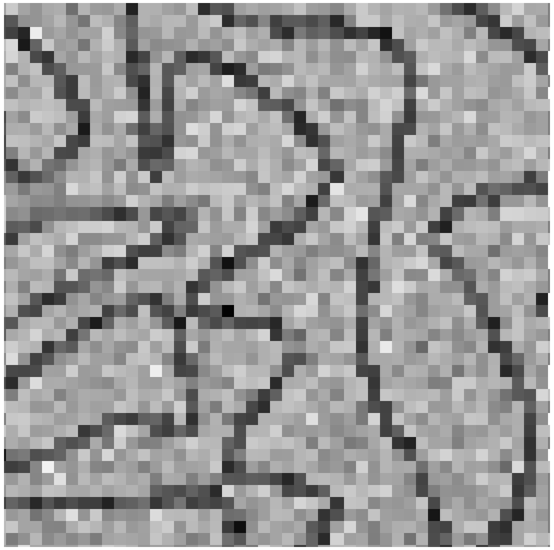

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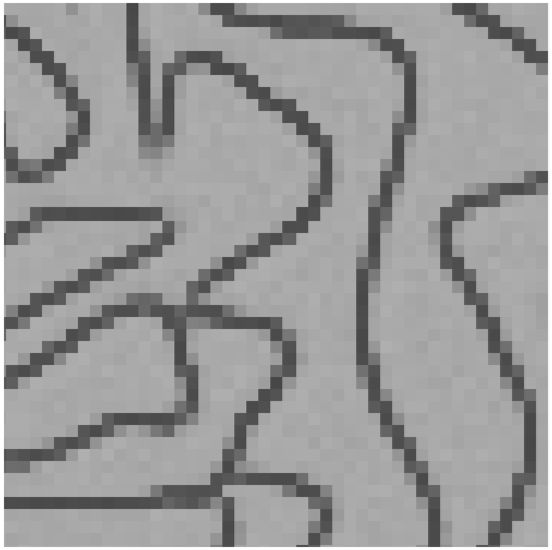

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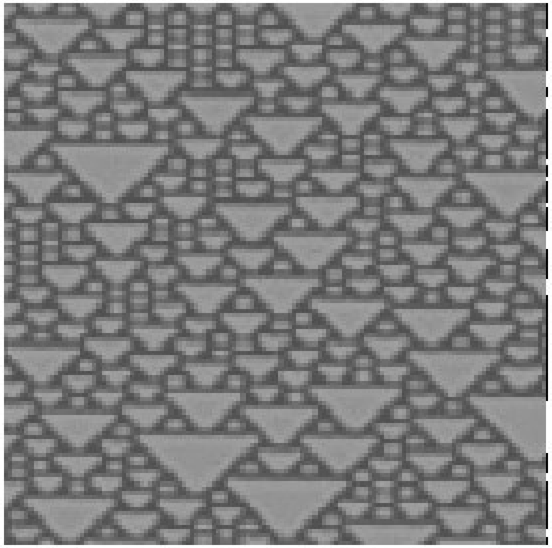
Results



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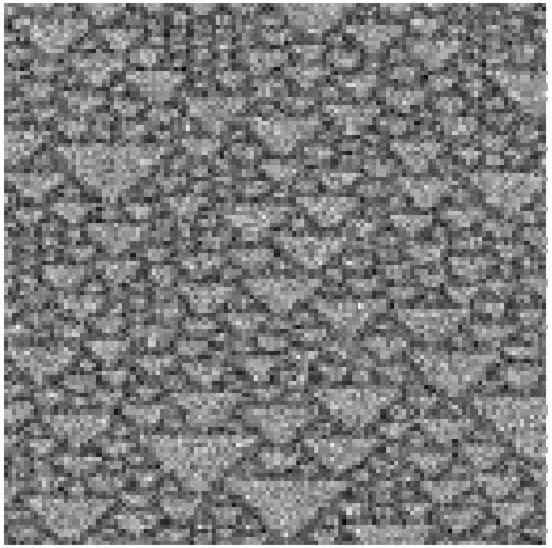

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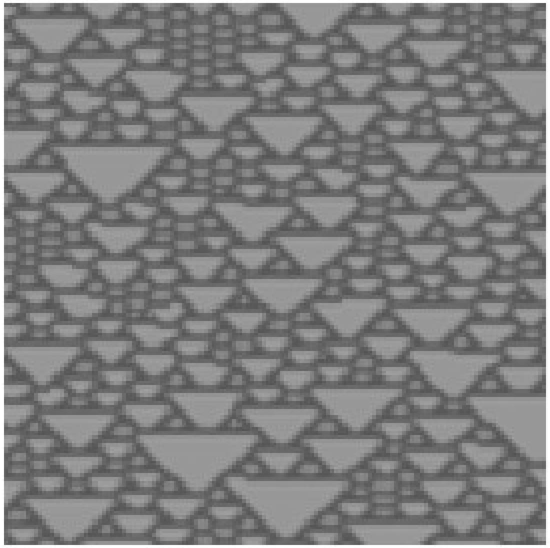
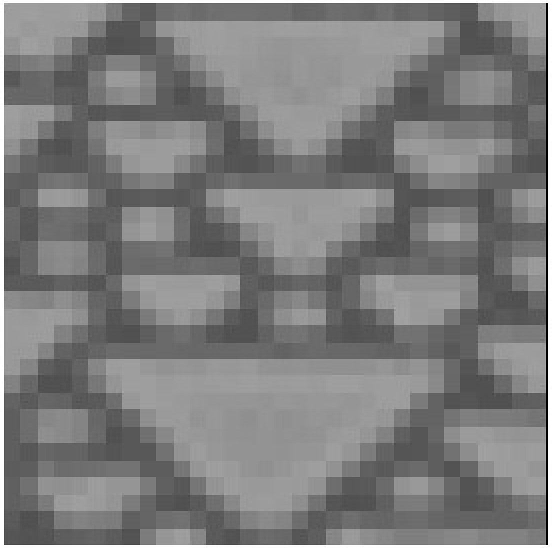
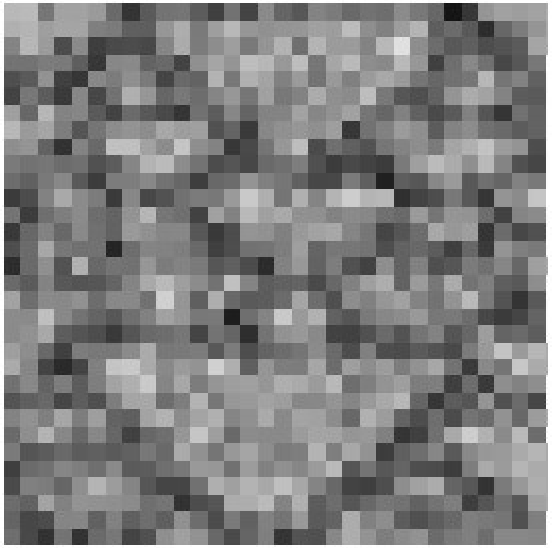


Filtered

Fractal



Original



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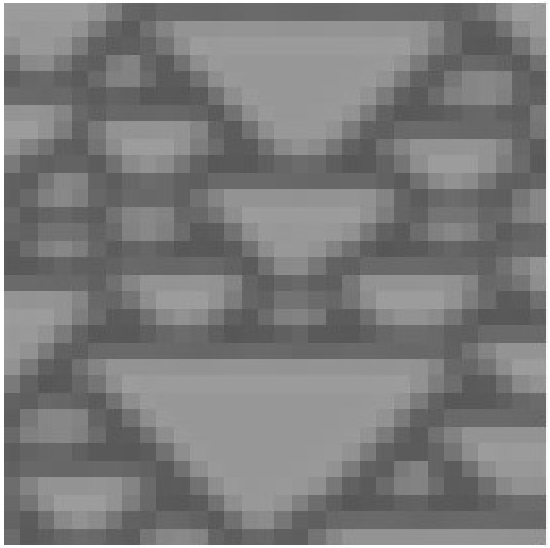
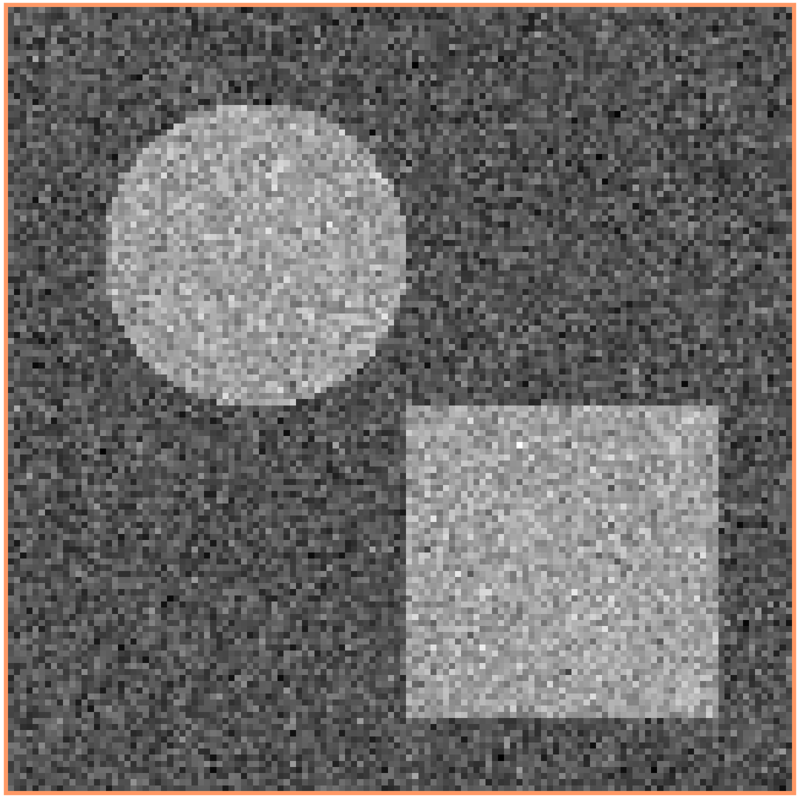


Filtered

Piecewise Constant

- Several 10s of Iterations
- Tends to obliterate rare events



Texture, Structure