3.1 Mathematical Background: Differential Geometry of Surfaces

We introduce the definition of the first and second fundamental forms of a 3D surface, and then derive the surface curvatures from them [do Carmo 1976; Lipschutz 1969].

3.1.1 First and Second Fundamental Differential Forms

A regular surface S in $\mathbf{E}^3$ (3D Euclidian space) is explicitly defined with respect to a known coordinate system by the following parametric representation.

$$S = \left\{ (x(u,v), y(u,v), z(u,v)) : \; (u,v) \in D \subseteq \mathbf{E}^2 \right\} \tag{1}$$

In this paper, we assume that a depth map from a single view is provided in the form of a digital graph surface (Monge patch surface). Thus the definition of the surface S can be simplified to:

$$S = \left\{ (x, y, z(x,y)) : \; (x,y) \in D \subseteq \mathbf{E}^2 \right\} \tag{2}$$

where $z(x,y)$ is meant to be a depth or range value at a point (x,y) in a given range image. Let the vector $(x, y, z(x,y))$ be denoted by $\mathbf{s}$ in the following discussion.

The surface S is uniquely determined by the first and second fundamental differential forms, which are referred to as **I** and **II** respectively. The first fundamental form **I** is defined by the following equation.

$$\mathbf{I} = d\mathbf{s} \cdot d\mathbf{s} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{3}$$

where the elements of the matrix $[\,g\,]$, which is called the first fundamental form matrix, are defined as:

$$g_{11} = E = \mathbf{s}_x \cdot \mathbf{s}_x = \frac{\partial \mathbf{s}}{\partial x} \cdot \frac{\partial \mathbf{s}}{\partial x} = 1 + \left(\frac{\partial z}{\partial x} \right)^2 \tag{4}$$

$$g_{12} = g_{21} = F = \mathbf{s}_x \cdot \mathbf{s}_y = \frac{\partial \mathbf{s}}{\partial x} \cdot \frac{\partial \mathbf{s}}{\partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \tag{5}$$

$$g_{22} = G = \mathbf{s}_y \cdot \mathbf{s}_y = \frac{\partial \mathbf{s}}{\partial y} \cdot \frac{\partial \mathbf{s}}{\partial y} = 1 + \left(\frac{\partial z}{\partial y} \right)^2 \tag{6}$$

Similarly, the second fundamental form **II** is given as:

$$\mathbf{II} = -d\mathbf{s} \cdot d\mathbf{n} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \tag{7}$$

where $\mathbf{n}$ denotes a unit normal vector which is defined to be

$$\begin{aligned} \mathbf{n} &= \frac{\frac{\partial \mathbf{s}}{\partial x} \times \frac{\partial \mathbf{s}}{\partial y}}{\left| \frac{\partial \mathbf{s}}{\partial x} \times \frac{\partial \mathbf{s}}{\partial y} \right|} \\ &= \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) \end{aligned} \tag{8}$$

and the second fundamental form matrix $[\ b\ ]$ is specified by the following equations:

$$b_{11} = L = \mathbf{s}_{xx} \cdot \mathbf{n} = \frac{\partial^2 \mathbf{s}}{\partial x^2} \cdot \mathbf{n} = \frac{\frac{\partial^2 z}{\partial x^2}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{9}$$

$$b_{12} = b_{21} = M = \mathbf{s}_{xy} \cdot \mathbf{n} = \frac{\partial^2 \mathbf{s}}{\partial x\,\partial y} \cdot \mathbf{n} = \frac{\frac{\partial^2 z}{\partial x\,\partial y}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{10}$$

$$b_{22} = N = \mathbf{s}_{yy} \cdot \mathbf{n} = \frac{\partial^2 \mathbf{s}}{\partial y^2} \cdot \mathbf{n} = \frac{\frac{\partial^2 z}{\partial y^2}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} \tag{11}$$

3.1.2 Surface Curvatures

We give the definition of the Gaussian and mean curvatures, and the principal curvatures and directions, as well as the relationships among them.

The Gaussian curvature K of a surface is defined from the first and second fundamental form matrices as follows:

$$\begin{aligned} K = det([g]^{-1})\, det([b]) &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2}{\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^2} \end{aligned} \tag{12}$$

The mean curvature H of a surface is similarly defined as:

$$\begin{aligned} H = \frac{1}{2} tr([g]^{-1}[b]) &= \frac{EN + GL - 2FM}{2(EG - F^2)} \\ &= \frac{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} \cdot \left(\frac{\partial z}{\partial y}\right)^2 + \frac{\partial^2 z}{\partial y^2} \cdot \left(\frac{\partial z}{\partial x}\right)^2 - 2\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial^2 z}{\partial x\,\partial y}}{2\left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{3/2}} \end{aligned} \tag{13}$$

The so-called principal curvatures k_1 and k_2 ($k_1 \geq k_2$) can be defined as the two roots of the following quadratic equation.

$$k^2 - 2Hk + K = 0 \tag{14}$$

Thus, k_1 and k_2 are obtained with respect to K and H as:

$$k_1 = H + \sqrt{H^2 - K} \tag{15}$$

$$k_2 = H - \sqrt{H^2 - K} \tag{16}$$

The Gaussian and mean curvatures can then be represented in terms of the two principal curvatures as follows:

$$K = k_1 \cdot k_2, \quad H = \frac{k_1 + k_2}{2} \tag{17}$$

The principal curvatures can also be determined as the extrema of the normal curvature function, which is defined as the ratio of two fundamental forms **II**/**I**. It should be noted that the principal curvatures are associated with specific directions, called the principal directions. These are determined for non-umbilic points ($k_1 \neq k_2$) by solving the following equation:

$$(EM - FL)\, dx^2 + (EN - GL)\, dx\, dy + (FN - GM)\, dy^2 = 0 \tag{18}$$

The principal directions can be shown to be orthogonal.

Surface curvatures are local characteristics which possess desirable invariance properties, including view independency [Besl & Jain 1986a]. This implies that they can be used for characterizing a surface in situations where it is partially occluded by others. In the following, local surface shape descriptors for range image segmentation are derived from the curvatures.

3.2 Interpreting Surface Curvatures

Gaussian curvature is an intrinsic surface property which refers to an isometric invariant of a surface [do Carmo 1976; Besl & Jain 1986a]. Both Gaussian and mean curvatures have the attractive characteristics of translational and rotational invariance; that is, at a point on a surface, these curvatures are invariant to translation and rotation of the object surface as long as the surface is visible.

Differential geometry tells us that an individual point on a surface can be locally classified into one of the following three surface types according to the sign of Gaussian curvature:
(1) $K > 0$ $\cdots$ *elliptic surface point*;
(2) $K < 0$ $\cdots$ *hyperbolic surface point*;
(3) $K = 0$ $\cdots$ *parabolic surface point*.
Fig. 3 illustrates the typical surface shape for these cases. In the case of $K = 0$, the surface is also referred to as a *developable surface*.

Another categorization of a surface is possible according to the local uniformity of Gaussian curvature. If the Gaussian curvature is constant on a surface, the surface is referred to as an *umbilic surface*. This includes the two cases: $k_1 = k_2 \neq 0$ (or $K = H^2 \neq 0$); $k_1 = k_2 = 0$ (or $K = H^2 = 0$) . In the former case, the surface is spherical, while the latter implies a planar surface, whose point is called a *planar umbilic* or *flat point*. This classification is implemented together with the above Gaussian curvature based classification for characterizing 3D object

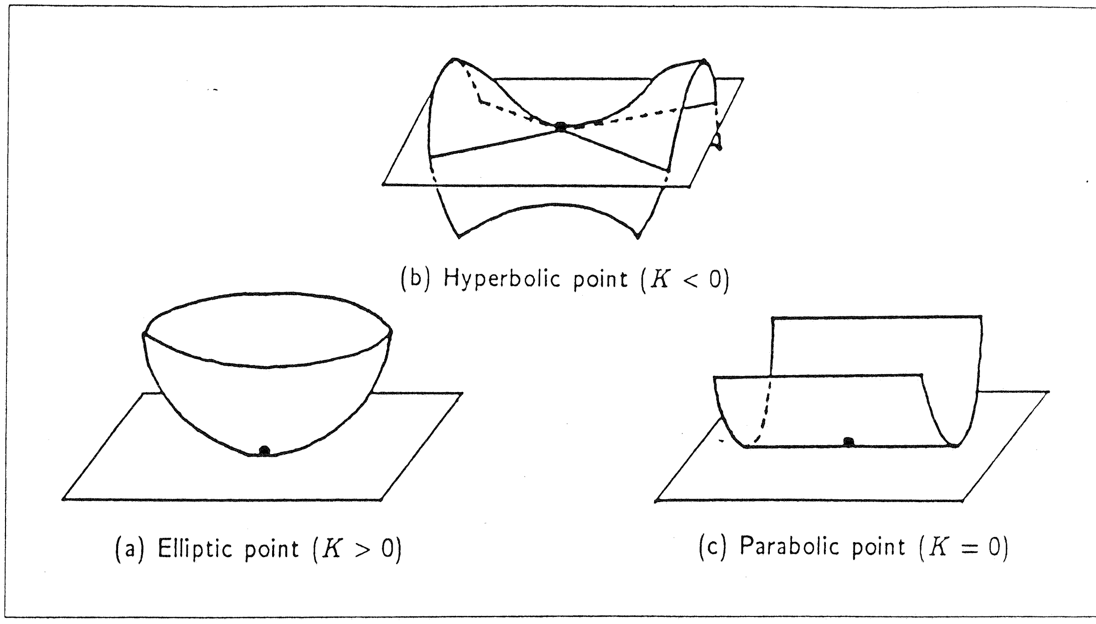

Figure 3 Shape of a surface in the vicinity of an elliptic, hyperbolic, and parabolic point.

shapes from range data in [Vemuri et al 1986; Verumi & Aggarwal 1987]. It provides a set of view-independent surface primitives for an entire surface of an object. However it is not powerful enough to describe a visible surface from a single view. For example, the convexity and concavity of a surface, which is thought to be important for describing an object, cannot be discriminated.

The sign of the mean curvature itself does not yield a good descriptor for surface shape. However, it does characterize surface shape at individual surface points if it is considered together with Gaussian curvature, while Gaussian curvature does so by itself as shown in Fig. 3. Besl and Jain have pointed out that the signs of Gaussian and mean curvatures yield a set of eight surface primitives which possess desirable invariance properties including view-independency, and are powerful enough to describe visible surfaces [Besl & Jain 1986a]. Given a coordinate system in which the z axis is directed toward the viewer, eight surface primitives are defined as follows:

(1) $K > 0$ and $H < 0$ $\cdots\cdot$ *peak surface*;
(2) $K > 0$ and $H > 0$ $\cdots\cdot$ *pit surface*;
(3) $K = 0$ and $H < 0$ $\cdots\cdot$ *ridge surface*;
(4) $K = 0$ and $H > 0$ $\cdots\cdot$ *valley surface*;
(5) $K = 0$ and $H = 0$ $\cdots\cdot$ *flat surface*;
(6) $K < 0$ and $H = 0$ $\cdots\cdot$ *minimal surface*;
(7) $K < 0$ and $H < 0$ $\cdots\cdot$ *saddle ridge surface*;
(8) $K < 0$ and $H > 0$ $\cdots\cdot$ *saddle valley surface*.

From the definition of surface curvatures, it is obvious that the case of $K > 0$ and $H = 0$ does not occur. Fig. 4 depicts the elemental shapes of these surface types. In a later section, they will be employed as surface primitives into which a range image is segmented.

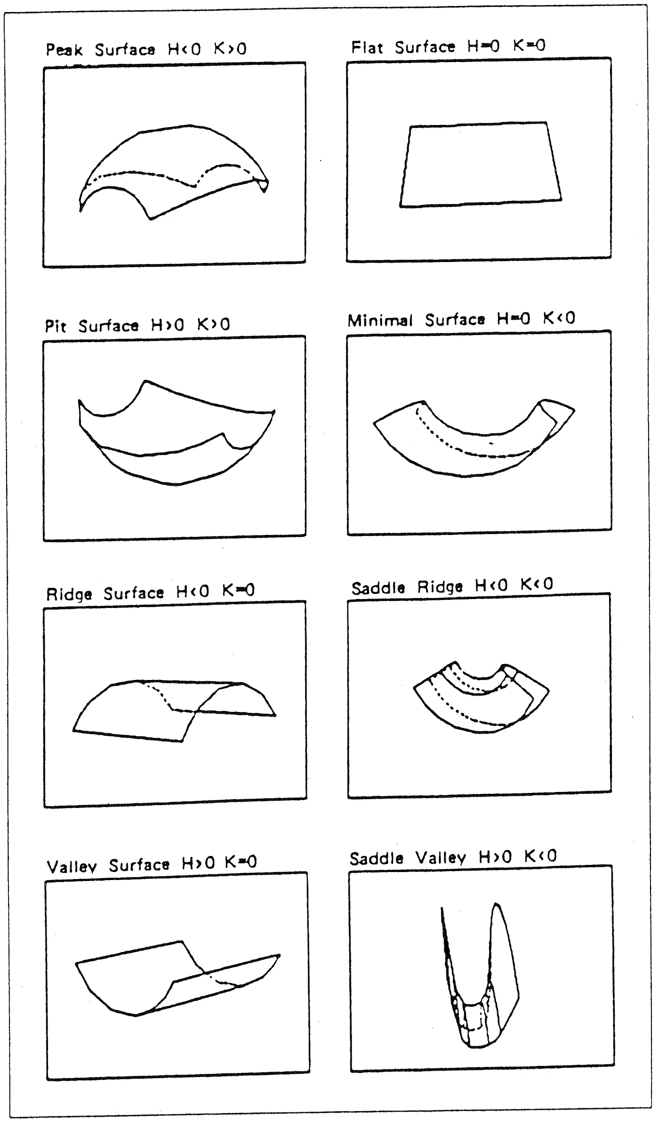

Figure 4 A set of eight view-independent surface types for a visible surface. (From [Besl & Jain 1986a].)

4. Computing Differential Geometric Properties of Piecewise Smooth Surfaces

This section proposes a technique for locally estimating differential geometric properties of a surface, especially for computing Gaussian and mean curvatures. It should be noted that differential geometry is a theory of smooth differentiable surfaces. At least second order differentiability is needed for computing surface curvatures on a surface, as is obvious from the definition of curvatures in the previous section. However, objects are usually not entirely smooth over all their surfaces, but are piecewise smooth. The problem is how to accurately estimate curvature properties for piecewise smooth surfaces. The proposed method is composed of three steps: (1) the analytical surface fit to a local window centered at an individual point; (2) the determination of the best window orientation for each point; (3) the compu-