week 8 Geometry

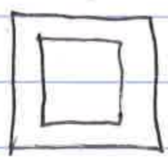
important in:

* correction of distortions in camera
* analysis of shape
* registration of images (e.g., a mosaic)

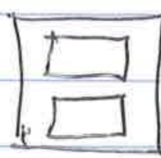
Shape

$$\underbrace{\text{set of points}}_{\text{segmentation}} \longrightarrow \underbrace{\text{features}}_{\text{shape representation}} \longrightarrow \begin{cases}\text{classification}\\ \text{comparison}\end{cases}$$

boundary may not be sufficient:



vs.



initial shape vector: points along boundary

$$\bar{x} = [x_1\ y_1\ x_2\ y_2 \ldots x_N\ y_N] \equiv \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_N & y_N \end{bmatrix}$$

$\Rightarrow$ are points organized? e.g., neighbors

Transforms: $T: \mathbb{R}^2 \to \mathbb{R}^2$

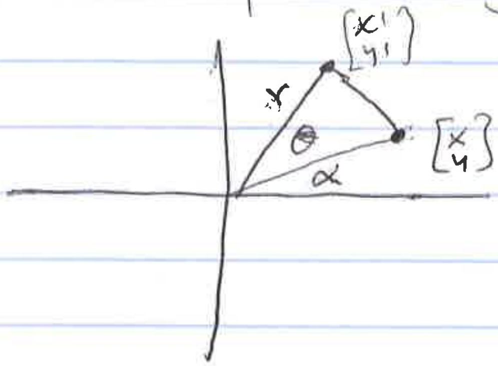
rigid transforms: translation, rotation

shape-preserving transforms: translation, rotation, scaling

Homogeneous coordinates

$$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ y_1 & y_2 & & y_N \end{bmatrix}$$ shape vector

can transform using 2×2 matrix T



e.g., rotation

$$x = r\cos\alpha$$
$$y = r\sin\alpha$$

$$x' = r\cos(\alpha+\theta)$$
$$x' = r[\cos\alpha\cos\theta - \sin\alpha\sin\theta]$$
$$= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$= x\cos\theta - y\sin\theta$$

$$y' = r\sin(\alpha+\theta)$$
$$y' = r[\sin\alpha\cos\theta + \cos\alpha\sin\theta]$$
$$= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$$
$$= y\cos\theta + x\sin\theta$$

2 equations put in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To transform set of points

$$S' = TS$$
$$2\times N = 2\times 2 * 2\times N$$

There are advantages in expressing transforms as matrix multiplication

but: addition is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Add a 'homogeneous' coordinate:

$$\begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Generally,

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

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$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = \alpha_{11} x + \alpha_{12} y + \alpha_{13}$$

$$y' = \alpha_{21} x + \alpha_{22} y + \alpha_{23}$$

Affine Transformation

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Should know Table 7.1 (p.174)
its correct version

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To translate and rotate:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Procrustes Transform</u> : preserves shape

$$T = \begin{bmatrix} \alpha & \gamma & \lambda_1 \\ -\gamma & \alpha & \lambda_2 \\ 0 & 0 & 1 \end{bmatrix}$$

(compare to)

<u>Procrustes alignment</u>

Find translation, rotation, scaling that minimizes:

$$\sum \text{dist}^2(\underset{\text{shape}}{S_i} - \underset{\text{reference}}{R_i})$$

Procrustes translation part of transform

Suppose the translation part is $\bar{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

Given 2 points $\bar{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ $\bar{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$

then $\text{dist}^2(\bar{x}_1 - x_2) \equiv$ length of vector squared

length of vector squared is $(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2$

rewrite as matrix (vector) product:

$$[x_{11} - x_{21}, x_{12} - x_{22}] \begin{bmatrix} x_{11} - x_{21} \\ x_{12} - x_{22} \end{bmatrix}$$

$$= (\bar{x}_1 - \bar{x}_2)^T (\bar{x}_1 - \bar{x}_2)$$

Gives rise to Eqn 7.16

$$Q' = \sum_{i=1}^{N} [\bar{x}_i + \bar{t} - \bar{x}_i']^T [\bar{x}_i + \bar{t} - \bar{x}_i]$$

find $\bar{t}$ that minimizes Q

book claims:

$$\bar{t} = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i' - \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i$$

$$Q = \sum_{i=1}^{N} [\bar{x}_i + \bar{t} - \bar{x}_i']^T [\bar{x}_i + \bar{t} - \bar{x}_i']$$

Let's look at first component t_1 $\qquad \bar{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$

$$Q = \sum_{i=1}^{N} [x_{i,1} + t_1 - x'_{i,1}, \; x_{i,2} + t_2 - x'_{i,2}] \begin{bmatrix} x_{i,1} + t_1 - x'_{i,1} \\ x_{i,2} + t_2 - x'_{i,2} \end{bmatrix}$$

$$= \sum_{i=1}^{N} [x_{i,1} + t_1 - (x_{i,1} + t'_1), \; x_{i,2} + t_2 - (x_{i,2} + t'_2)] \begin{bmatrix} x_{i,1} + t_1 - (x_{i,1} + t'_1) \\ x_{i,2} + t_2 - (x_{i,2} + t'_2) \end{bmatrix}$$

$$= \sum_{i=1}^{N} [t_1 - t'_1, \; t_2 - t'_2] \begin{bmatrix} t_1 - t'_1 \\ t_2 - t'_2 \end{bmatrix}$$

$$Q = \sum_{i=1}^{N} (t_1 - t'_1)^2 + (t_2 - t'_2)^2$$

So, $\displaystyle \frac{\partial Q}{\partial t_1} = \sum_{i=1}^{N} 2(t_1 - t'_1)$

To minimize, set to 0 and solve

$$0 = \sum_{i=1}^{N} 2(t_1 - t'_1)$$

$$= \sum_{i=1}^{N} (t_1 - t'_1)$$

$$= N t_1 - \sum_{i=1}^{N} t'_1$$

$\Rightarrow$

$$N t_1 = \sum_{i=1}^{N} t'_1$$

$$t_1 = \frac{1}{N} \sum_{i=1}^{N} t'_1$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_{i,1} - \frac{1}{N} \sum_{i=1}^{N} x_{i,1} + \frac{1}{N} \sum_{i=1}^{N} t'_1 = \underbrace{\frac{1}{N} \sum_{i=1}^{N} x'_{i,1}}_{\langle y \rangle} - \underbrace{\frac{1}{N} \sum_{i=1}^{N} x_{i,1}}_{\langle x \rangle}$$

X, Y

$\bar{X} \to \bar{X} + \bar{t} = \bar{X}'$

$Y = \begin{pmatrix} s & \beta & a \\ \beta & s & b \\ 0 & 0 & 1 \end{pmatrix} [$

217,406 E

283,439

216,422

X Y

x_i x_i'

$\bar{x}_i \to \bar{x}_i + \bar{t} = \bar{x}_i'$

$\bar{t} = \text{mean}($

$$Q = \sum_{i=1}^{N} [s x_i - x_i']^T [s \bar{x}_i - \bar{x}_i']$$

$$\sum_{i=1}^{N} [s x_{i,1} - x'_{i,1}, s x_{i,2} - x'_{i,2}] \begin{bmatrix} s x_{i,1} - x'_{i,1} \\ s x_{i,2} - x'_{i,2} \end{bmatrix}$$

$$Q = \sum_{i=1}^{N} (s x_{i,1} - x'_{i,1})^2 + (s x_{i,2} - x'_{i,2})^2$$

$$\frac{\partial Q}{\partial s} = \sum_{i=1}^{N} [2(s x_{i,1} - x'_{i,1}) x_{i,1} + 2(s x_{i,2} - x'_{i,2}) x_{i,2}]$$

$$\sum_{i=1}^{N} \cancel{2}(s x_{i,1} - x'_{i,1}) x_{i,1} = -\sum_{i=1}^{N} \cancel{2}(s x_{i,2} - x'_{i,2}) x_{i,2}$$

$$\sum [s x_{i,1}^2 - x'_{i,1} x_{i,1}] = -\sum_{i=1}^{N} [s x_{i,2}^2 - x'_{i,2} x_{i,2}]$$

$$\sum s x_{i,1}^2 - \sum x'_{i,1} x_{i,1} = -\sum s x_{i,2}^2 + \sum x'_{i,2} x_{i,2}$$

$$\sum s x_{i,1}^2 + \sum s x_{i,2}^2 = \sum x'_{i,1} x_{i,1} + \sum x'_{i,2} x_{i,2}$$

$$s = \frac{\sum x'_{i,1} x_{i,1} + \sum x'_{i,2} x_{i,2}}{\sum x_{i,1}^2 + \sum x_{i,2}^2}$$

Assume reference shape has mean $\begin{bmatrix}0\\0\end{bmatrix}$

Then $\bar{t} = -\langle x \rangle$ mean of shape points

So, shift shape points centered at origin

Scale : same kind of analysis

orientation : use a sophisticated method
similar to principle axes method

Matlab provides : procrustes
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Projective Transform

consider object (3D) points as on a plane
mapped to an image plane

$$T = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$\bar{x}' = T * \bar{x}$$

$$\Rightarrow \quad x_1' = \alpha_{11} x_1 + \alpha_{12} y_1 + \alpha_{13}$$
$$y_1' = \alpha_{21} x_1 + \alpha_{22} y_1 + \alpha_{23}$$
$$1 = \alpha_{31} x_1 + \alpha_{32} y_2 + \alpha_{33}$$

Build a set of linear equations:

each point gives 3 eqns

$$\overset{b}{\begin{bmatrix} x_1' \\ y_1' \\ 1 \\ x_2' \\ y_2' \\ 1 \\ \vdots \end{bmatrix}} = \overset{A}{\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & y_2 & 1 \\ & & & & \vdots & & & & \end{bmatrix}} \overset{\bar{x}}{\begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix}}$$

unknown

$x = A \backslash b$

$T \leftarrow$ reshape x

Nonlinear Transform

$$x' = a_0 x^2 + a_1 xy + a_2 y^2 + a_3 x + a_4 y + a_5$$
$$y' = b_0 x^2 + b_1 xy + b_2 y^2 + b_3 x + b_4 y + b_5$$