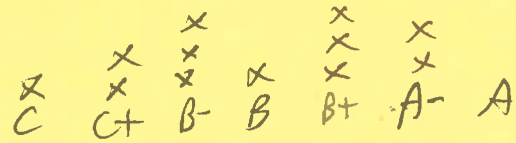
6\1

CS4640 Week 6    Fourier Transform

Quiz 4




You need to know how this calculus relates to image analysis

Q1: idea is to average neighborhoods on either side

Q2: Just plug in values to formula

Q3: gradient is a vector

A1: graded

A2: graded by Thursday (goal!)

A3: due on thursday

Q1. You are supposed to think about the problem and provide a carefully thought out solution.

E.g., performance measures

define ground truth as where all edge detectors agree?

not good (try it!)

how about look at a threshold for $|\nabla f|$ ∋

green + white edges show up

[dx, dy] = gradient( )

mag = sqrt(dx.^2 + dy.^2);

see CS4640-week6

## Q 2.

Go over class notes for this.

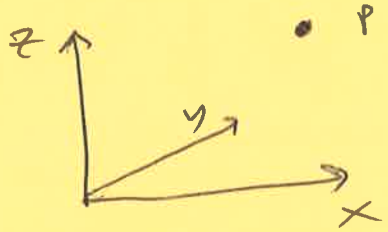
Look at cs4640_week6

## Q3 Look at linear features & rectangles

## Fourier Series

Fourier the Man!

Consider basis elements, e.g., for geometry




$\bar{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

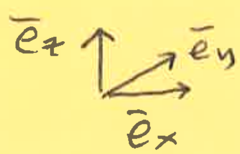
what does it mean?

* Go distance a along x
* Go distance b along y
* Go distance c along z

i.e., $\bar{p} = a\,\bar{e}_x + b\,\bar{e}_y + c\,\bar{e}_z$

where:

$\bar{e}_x, \bar{e}_y, \bar{e}_z$ are unit vectors + orthogonal




In fact,

$\bar{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \bar{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \bar{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

can view it as

$$\left.\begin{aligned} a &= \bar{p} \cdot \bar{e}_x = a \cdot 1 + b \cdot 0 + c \cdot 0 \\ b &= \bar{p} \cdot \bar{e}_y \\ c &= \bar{p} \cdot \bar{e}_z \end{aligned}\right\}$$

the coefficient is the projection of $\bar{p}$ onto unit basis vectors

A Fourier series represents a function, f, as a sum of basis functions:

basis functions

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos(2\pi n t) + b_n \sin(2\pi n t) \right)$$

signal

$a_n, b_n$ are coefficients or weights

n is the wave frequency

cs 4640_week6a

Note that:

$$\int_0^1 \sin(2\pi n t) \sin(2\pi m t)\, dt = 0 \qquad m \neq n$$

$$\int_0^1 \sin(2\pi n t) \sin(2\pi n t)\, dt = 1/2$$

$$\int_0^1 \cos(2\pi n t) \cos(2\pi m t)\, dt = 0$$

$$\int_0^1 \cos(2\pi n t) \cos(2\pi n t)\, dt = 1/2$$

$\sin(2\pi n t)$ with $\cos(2\pi n t)$

forms: unit orthogonal basis elements

Can show:

$$\int_0^1 f(t)\cos(2\pi n t)\,dt = \frac{a_n}{2}$$

$$\int_0^1 f(t)\sin(2\pi n t)\,dt = \frac{b_n}{2}$$

$$\int_0^1 f(t)\,dt = a_0$$

these are equivalent to continuous dot products

Try $f(t) = \sin(2\pi t)$

should be $b_1 = 1$ and $0 = b_n$, $n \neq 1$

$$b_n = 2\int_0^1 \sin(2\pi t)\sin(2\pi n t)\,dt$$

for $n \neq 1$, integral is 0

for $n = 1$, integral is $1/2$

$$b_1 = 2 \cdot \frac{1}{2} = 1$$

all $\int_0^1 \sin(2\pi \cdot 1 \cdot t)\cos(2\pi n t)\,dt$ are 0

CS 4640 - week6a

Consider step edge (square wave!)

$$f(t) = \begin{cases} 1 & 0 < t \le 1/2 \\ 0 & \frac{1}{2} < t < 1 \end{cases}$$

$$\text{so,} \quad b_n = 2 \int_0^{1/2} f(t) \sin(2\pi n t)\, dt \quad \leftarrow \text{why } 1/2\,?$$

$$b_n = \frac{-2\cos(2\pi n t)}{2\pi n} \Big|_0^{1/2}$$

$$= \frac{1 - \cos(\pi n t)}{\pi n}$$

when $n$ is odd, $\cos(\pi n t) = -1$

$$\text{So } b_n = \frac{2}{\pi n}$$

when $n$ is even, $\cos(\pi n t) = 1$

$$\text{so. } b_n = 0$$

Note: $a_0 = \int_0^{1/2} dt = \frac{1}{2}$

---

$$a_n = \frac{2\sin(2\pi n t)}{2\pi n} \Big|_0^{1/2}$$

$$= \frac{\sin(\pi n t)}{\pi n}$$

$= 0$ for all $n$

So, why are we interested in Fourier transforms?

moves from time domain or space domain } → frequency domain

frequencies have meaning
and we can manipulate them to
transform images! (point 3 below)

Key ideas

(1) harmonic (frequency) content of signals
frequency components

(2) Fourier representation is complete:
weights are called the spectrum

(3) Fourier processing: frequency components
are manipulated

(4) Space + Fourier domains are reciprocal:
high frequencies capture small-scale (distance) changes
large spatial homogeneity captured by low frequencies

(5) Fourier series: discrete frequencies
Fourier transform: continuous frequencies

6\7

Note that book (p. 118) has:

$$a_n = \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(x) \cos\left(\frac{2\pi n x}{\lambda}\right) dx$$

$$b_n = \frac{2}{\lambda} \int_{-\frac{\lambda}{2}}^{\lambda/2} f(x) \sin\left(\frac{2\pi n x}{\lambda}\right) dx$$

but any constant can be added to the limits of integration and still be over 1 period.

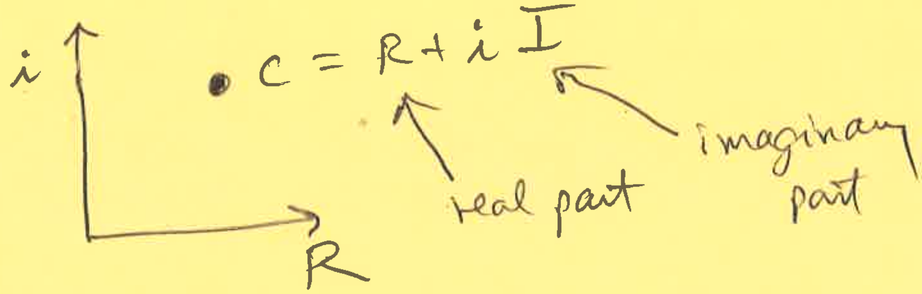
---

Complex numbers: why?

because $e^{j\theta} = \cos\theta + j\sin\theta$

see CS4640-week6

a complex number is in 2D (complex plane)




$R + I \in \mathbb{R}$

$i = \sqrt{-1}$ also called $j$

polar coordinates:

$$c = |c|(\cos\theta + i\sin\theta) = |c|e^{i\theta}$$

$|c| = \sqrt{R^2 + I^2}$ length of vector $c$

$\theta$ is the angle of $c$ (from the $R$ axis)

$$\theta = \text{atan2}(I, R)$$

$c^*$ is complex conjugate of $c$ $\quad c^* = R - iI$

complex functions

$$F(u) = R(u) + iI(u)$$

$$F^*(u) = R(u) - iI(u) \quad \text{conjugate}$$

$$|F(u)| = \sqrt{R(u)^2 + I(u)^2}$$

$$\theta(u) = \text{atan2}(I(u), R(u))$$

Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i2\pi n x}{\lambda}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n \exp(i k n x)$$

$$c_n = \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(x)\exp(i k n x)\,dx$$