

The Finite Difference Method for Transport

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April 18, 2016

1 Introduction

We will now discuss the application of finite differences to the linear transport equation or advection equation. This is a linear first-order PDE that describes the evolution of a scalar in a divergence-free velocity field. It is given by:

$$\frac{\partial Q}{\partial t} + u \frac{\partial q}{\partial x} = 0, \quad (1)$$

where $q(x, t)$ is the scalar quantity being transport and u is some constant velocity field. We will focus on the 1D version of this equation. This document will not derive methods as we did in class. Instead, we will summarize the different methods that we derived, provide the Von Neumann growth factors and the actual FD formulae for this equation.

2 Schemes and Growth factors

We discussed several methods in class. Their properties are summarized in the table below. In all the cases below, $\alpha = u \frac{\Delta t}{h}$.

Method	Order	$ G ^2$	Formula
FTCS	$O(\Delta t, h^2)$	$1 + \sin^2(kh)\alpha^2$	$Q_j^{n+1} = Q_j^n - \frac{\alpha}{2}(Q_{j+1}^n - Q_{j-1}^n)$
Lax Friedrichs	$O(\Delta t, h^2)$	$1 - \sin^2(kh)(1 - \alpha^2)$	$Q_j^{n+1} = \frac{Q_{j+1}^n + Q_{j-1}^n}{2} - \frac{\alpha}{2}(Q_{j+1}^n - Q_{j-1}^n)$
Upwind	$O(\Delta t, h)$	$1 - 2 \alpha (1 - \alpha)\cos(kh)$	$Q_j^{n+1} = Q_j^n \mp (Q_{j\pm 1}^n - Q_j^n)$
Leapfrog	$O(\Delta t^2, h^2)$	1	$Q_j^{n+1} = Q_j^{n-1} - \alpha(Q_{j+1}^n - Q_{j-1}^n)$
BTCS	$O(\Delta t, h^2)$	Unconditionally stable	$Q_j^{n+1} = Q_j^n + \frac{\alpha}{2}(Q_{j+1}^{n+1} - Q_{j-1}^{n+1})$

3 Modified Equation

While the transport equation as written above contains no dissipation or dispersion, any numerical method will introduce some of each. One technique to analyze how much dissipation and dispersion a numerical method introduces is to write out the *modified equation* for the schemes. We will work this out for Lax-Friedrichs, and leave the rest as an exercise.

$$\frac{Q_j^{n+1} - \frac{1}{2}(Q_{j-1}^n + Q_{j+1}^n)}{\Delta t} + u \frac{Q_{j+1}^n - Q_{j-1}^n}{2h} = 0, \quad (2)$$

$$\implies \frac{Q_j^{n+1} - Q_j^n}{\Delta t} + u \frac{Q_{j+1}^n - Q_{j-1}^n}{2h} - \frac{Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n}{2\Delta t} = 0. \quad (3)$$

Taylor expanding the Q^{n+1} term in time and the Q_{j-1} and Q_{j+1} terms in space, we get

$$q_t + \frac{1}{2}q_{tt}\Delta t + uq_x + \frac{1}{6}uq_{xxx}h^2 - \frac{1}{2\Delta t}h^2 - \frac{1}{24\Delta t}q_{xxxx}h^4 + \dots, \quad (4)$$

$$= (q_t + uq_x) + \frac{1}{2}(q_{tt}\Delta t - q_{xx}\frac{h^2}{\Delta t}) + \dots, \quad (5)$$

$$= (q_t + uq_x) + \frac{1}{2}(u^2\Delta t - \frac{h^2}{\Delta t}q_{xx}), \quad (6)$$

where we have used $q_{tt} = u^2q_{xx}$, obtained by differentiating the transport equation. Thus, the modified equation is

$$q_t + uq_x = \frac{h^2}{2\Delta t}(1 - \alpha^2)q_{xx}. \quad (7)$$

Thus, when we apply Lax-Friedrichs to the advection equation, we are actually solving an advection-diffusion equation! This is the numerical diffusion introduced by Lax-Friedrichs. If we had retained the q_{xxx} term, we would have seen that it is multiplied by h^2 . This is a numerical dispersion term. One can apply similar techniques to the Upwind, Leapfrog and BTCS methods as well. Typically, first order schemes are diffusive, while higher-order methods are dispersive.