

Practice Exam 1 Name and Unid: _____

1. Suppose a laminar object with boundaries $y = x$ and $y = \sqrt{x}$ has a density function $\rho(x, y) = y(1 - x^2/2)$. Find the mass of the object.

Solution:

$$\int_0^1 \int_x^{\sqrt{x}} y(1 - x^2/2) dy dx = \text{youcandothis...}$$

2. Suppose a factory must perform two operations on items they are producing. The time it takes to operation 1 and 2 are $x \geq 0$ and $y \geq 0$, respectively, with a joint probability distribution $\rho(x, y) = 4xye^{-x^2-y^2}$. What is the probability that the time for either x or y operations is more than 2?

Solution: first check it integrates to 1 over the whole domain

$$\begin{aligned} 4 \int_0^\infty \int_0^\infty xye^{-x^2-y^2} dx dy &= 4 \int_0^\infty xe^{-x^2} dx \int_0^\infty ye^{-y^2} dy \\ &\quad u = x^2, \quad du/dx = 2x \\ \int_0^\infty xe^{-x^2} dx &= \frac{1}{2} \int_0^\infty e^{-u} du = \frac{1}{2} \\ \implies 4 \int_0^\infty \int_0^\infty xye^{-x^2-y^2} dx dy &= 1 \end{aligned}$$

The set D is comprised of three subsets $D_1 = \{(x, y) | x > 2, y > 2\}$, $D_2 = \{(x, y) | x > 2, 0 < y < 2\}$, and $D_3 = \{(x, y) | y > 2, 0 < x < 2\}$, so that $D = D_1 \cup D_2 \cup D_3$ and $P(D) = P(D_1) + P(D_2) + P(D_3)$

$$\begin{aligned}
 P(D_1) &= P(x > 2, y > 2) = 4 \int_2^\infty \int_2^\infty xy e^{-x^2-y^2} dx dy \\
 &\quad \frac{1}{2} \int_2^\infty e^{-u} du = \frac{e^{-2}}{2} \\
 &\implies P(D_1) = 4 \frac{e^{-4}}{4} = e^{-4}
 \end{aligned}$$

$$\begin{aligned}
 P(D_2) &= P(x > 2, 0 < y < 2) = 4 \int_0^2 \int_2^\infty xy e^{-x^2-y^2} dx dy \\
 &= 4 \left(\int_2^\infty x e^{-x^2} dx \right) \left(\int_0^2 y e^{-y^2} dy \right) \\
 &= 4 \frac{e^{-2}}{4} (1 - e^{-2}) \\
 P(D_3) &= P(D_2) \\
 &= e^{-2} (1 - e^{-2}) = e^{-2} - e^{-4} \\
 \implies P(D) &= 2e^{-2} - 2e^{-4} + e^{-4} = 2e^{-2} - e^{-4}
 \end{aligned}$$

Another way to check that this is right is to note that

$$\begin{aligned}
 P(D) &= 1 - P(x < 2, y < 2) = 1 - 4 \left(\int_0^2 y e^{-y^2} dy \right)^2 \\
 &= 1 - (1 - e^{-2})^2 = 1 - (1 - 2e^{-2} + e^{-4}) = 2e^{-2} - e^{-4}
 \end{aligned}$$

3. Using polar coordinates evaluate $\iint_{\Omega} f dA$, where Ω is the circle of radius 2 and $f(r, \theta) = r^2 \cos(2\theta)$.

Solution:

$$\begin{aligned}
 \iint_{\Omega} f dA &= \int_0^{2\pi} \int_0^2 r^3 \cos(2\theta) dr d\theta \\
 &= 2^4/4 \int_0^{2\pi} \cos(2\theta) d\theta = 2^4/4 \sin(2\theta) \Big|_0^{2\pi} = 0
 \end{aligned}$$

4. The concentration of perfume is given by $f(x, y, z) = e^{-(x^2+y^2+z^2)}(1 + \cos(x^2 + y^2 + z^2))$. Find the total mass of perfume inside the ball B of radius 10 meters. Set up the appropriate iterated integral but do not solve.

Solution: Use spherical coordinates: $x = \rho \sin(\phi) \cos \theta$, $y = \rho \sin(\phi) \sin \theta$, $z = \rho \cos(\phi)$ and $\rho^2 = x^2 + y^2 + z^2$. $dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$

$$Mass = \iiint_B f dV = \int_0^{2\pi} \int_0^\pi \int_0^{10} e^{-\rho^2} (1 + \cos(\rho^2)) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

5. Consider S the surface $z = \sin(x + y)$ on the unit square.

(a) Find a (x, y) -parametrization of S

Solution: Parametrize S : $\vec{r}(x, y) = [x, y, \sin(x + y)]^t$.
Range: $D = \{(x, y) | x \in [0, 1] \text{ and } y \in [0, 1]\}$.

(b) Find the normal vector $\vec{N}(x, y) = \vec{r}_\phi \times \vec{r}_\theta$.

Solution: Find $\vec{r}_x, \vec{r}_y$:

$$\vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ \cos(x + y) \end{pmatrix}, \quad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ \cos(x + y) \end{pmatrix}$$

Find $\vec{N}$:

$$\vec{N}(x, y) = \vec{r}_\phi \times \vec{r}_\theta = \begin{pmatrix} i & j & k \\ 1 & 0 & \cos(x + y) \\ 0 & 1 & \cos(x + y) \end{pmatrix} = \begin{pmatrix} -\cos(x + y) \\ -\cos(x + y) \\ 1 \end{pmatrix}.$$

magnitude:

$$\|\vec{N}(x, y)\| = \sqrt{2 \cos^2(x + y) + 1} = \sqrt{\cos(2(x + y)) + 2}$$

(c) Set up the surface area iterated integral but do not solve:

Solution:

$$area = \int_S dS = \int_0^1 \int_0^1 \sqrt{\cos(2(x + y)) + 2} \, dx \, dy$$

6. Integrate $\iint_E \frac{2x+y}{x+3y} dA$ over the set E bounded by the lines $y = -x/3 + 1$, $y = -x/3 + 2/3$, and $y = -2x$ and $y = -2x + 1$

Solution:

$$\begin{aligned}
u &= 2x + y \\
v &= x + 3y \\
J^{-1} &= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \\
||J^{-1}|| &= 5 \implies ||J|| = 1/5 \\
J &= \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \\
D &= \{(u, v) | u \in [0, 1], v \in [2, 3]\} \iint_E \frac{2x + y}{x + 3y} dA = \iint_D \frac{u}{v} \frac{1}{5} dA \\
&= \frac{1}{5} \int_0^1 \int_2^3 \frac{u}{v} dv du = \frac{1}{10} (\ln(3) - \ln(2))
\end{aligned}$$

7. Compute the gradient field ∇f given the potential function $f(x, y) = 1 - x^2 + y^2$

Solution:

$$\nabla f = [-2x, 2y]^T$$

8. Compute the work integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = [x - y, x + y]^T$ where C is defined by the parameterization $\vec{r} = [t, t - t^2]^T$ from $t = 0$ to $t = 1$

Solution:

$$\begin{aligned}
\vec{r}' &= [1, 1 - 2t]^T \\
\vec{F}(t) &= [t - (t - t^2), t + t - t^2]^T = [t^2, 2t - t^2]^T \\
\int_0^1 [t^2, 2t - t^2]^T \cdot [1, 1 - 2t]^T dt &= \int_0^1 t^2 + (2t - t^2)(1 - 2t) dt \\
&= \int_0^1 t^2 + 2t - t^2 - 4t^2 - 2t^3 dt \\
&= \int_0^1 2t - 4t^2 - 2t^3 dt \\
&= [1 - 4/3 - 1/2]
\end{aligned}$$

9. Determine if the vector field $\vec{F} = [x - y, x + y]^T$ is conservative on the disk centered at the origin with radius 10.

Solution:

$$P_y = -1$$

$$Q_x = 1$$

Not conservative.

10. Determine if the vector field

$$\vec{F} = \left[\frac{x-y}{(x^2+y^2)}, \frac{x+y}{(x^2+y^2)} \right]^T$$

is conservative on (a) the disk centered at the origin with radius 10, and (b) the disk centered at (3,3) with radius 2.

Solution:

$$P_y = \frac{-(x^2+y^2) - 2y(x-y)}{(x^2+y^2)^2} = \frac{-x^2+y^2-2xy}{(x^2+y^2)^2} Q_x = \frac{(x^2+y^2) - 2x(x+y)}{(x^2+y^2)^2} = \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}$$

(a) Not conservative due to singularity (b) Conservative

11. Determine if a potential function
- f
- exists for
- $\vec{F} = [y^2, 2xy + 1]^T$
- , and if so, find
- f
- .

Solution: The vector field is smooth on all of the plane and $P_y = 2y = Q_x$, so a potential function exists:

$$\begin{aligned} \int P dx &= xy^2 + C_1(y) \\ \int Q dy &= xy^2 + y + C_2(x) \\ \implies C_1(y) &= y \quad C_2 = C \\ f(x, y) &= xy^2 + y \\ \text{check: } \nabla f &= \vec{F} = [y^2, 2xy + 1]^T \end{aligned}$$