

Math 2250-001 Fall 2016 Midterm 1

Name and uID: _____

1. (a) (2 points) Assuming k and C are two given constants, directly write the solution of the following IVP.

$$\frac{dx}{dt} = kx, \quad x(0) = C.$$

Solution: Either by inspection or by separation of variables, the solution is

$$x(t) = Ce^{kt}.$$

- (b) (1 point) If $k > 0$ and $C > 0$, what is $\lim_{t \rightarrow \infty} x(t)$?

Solution: $\lim_{t \rightarrow \infty} x(t) = +\infty$.

- (c) (1 point) If $k < 0$ and $C < 0$, what is $\lim_{t \rightarrow \infty} x(t)$?

Solution: $\lim_{t \rightarrow \infty} x(t) = 0$.

- (d) (1 point) If $k > 0$ and $C < 0$, what is $\lim_{t \rightarrow \infty} x(t)$?

Solution: $\lim_{t \rightarrow \infty} x(t) = -\infty$.

2. (5 points) **Without solving the DE**, determine whether the solution to the IVP

$$yy' = x - 1, \quad y(1) = 0$$

exist and is unique.

HINT: Check the hypothesis of the existence and uniqueness theorem.

Solution: From above, the y -derivative of f is discontinuous at $(1, 0)$, thus existence and uniqueness is not guaranteed. Using separation of variables, we can solve for a solution $y = \sqrt{x^2 - 2x + 1} = x - 1$; however, $y = 0$ is also a solution. Therefore, we see there are two solutions that both pass through $(1, 0)$ which verifies that uniqueness is not satisfied.

3. (5 points) Solve the following initial value problem. $y' = x - y$, $y(0) = 1$.

Solution: This is a linear first order differential equation. Hence, we first multiply by the integrating factor to obtain

$$e^x y' + e^x y = e^x x$$
$$\frac{d}{dx}(e^x y) = e^x x.$$

Integrating both sides and using integration by parts, we find that

$$y(x) = x - 1 + C e^{-x}.$$

Using the initial condition, we find that the solution is

$$y(x) = x - 1 + 2e^{-x}.$$

4. (5 points) A roast is removed from a 350° F oven and left to cool at room temperature, which is 70° F. After 30 minutes, the temperature of the roast is 210° F. Let $x(t)$ be the temperature of the roast t minutes after it is taken out of the oven. Suppose $x(t)$ evolves according to Newton's law of cooling

$$\frac{dx}{dt} = k(70 - x).$$

Find an exact formula for k .

Solution: Separating variables and integrating yields

$$\ln |70 - x| = -kt - C.$$

Hence,

$$x(t) = 70 - Ce^{-kt}.$$

Since the roast was removed from 350° F oven, we have that $x(0) = 350$ and thus

$$x(t) = 70 + 280e^{-kt}.$$

Since the roast is 210° F after being out of the oven for 30 minutes, we have that

$$x(30) = 70 + 280e^{-30k} = 210.$$

Solving for k in the above equation, we conclude that

$$k = \frac{-\ln(\frac{1}{2})}{30}.$$

5. (5 points) Find all the equilibrium solutions to the following differential equation **and** indicate their stability.

$$x'(t) = x^4 - 3x^3 + 2x^2$$

Solution: First, we factor the righthand side of the differential equation to obtain

$$x'(t) = x^2(x - 1)(x - 2).$$

Thus, the equilibrium solutions are

$$x(t) = 0, \quad x(t) = 1, \quad x(t) = 2.$$

Since $x' > 0$ if $0 < x < 1$, we have that

$$x(t) = 0 \text{ is unstable.}$$

Since $x' < 0$ if $1 < x < 2$ and $x' > 0$ if $0 < x < 1$, we have that

$$x(t) = 1 \text{ is stable.}$$

Since $x' > 0$ if $x > 2$, we have that

$$x(t) = 2 \text{ is unstable.}$$

6. (5 points) The differential equation

$$\frac{dx}{dt} = \frac{1}{12}x(12 - x) - h$$

models a logistic population with harvesting at rate h . Determine the dependence of the number of critical points on the parameter h .

Hint: Recall the quadratic formula. If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solution: Using the quadratic formula on the right hand side of the differential equation, we see that $\frac{dx}{dt} = 0$ if and only if

$$x = \frac{-1 \pm \sqrt{1 - \frac{4}{12}h}}{-\frac{2}{12}} = 6 \pm \sqrt{1 - \frac{1}{3}h}.$$

Thus, there are

two critical points if $h < 3$

one critical points if $h = 3$

no critical points if $h > 3$.

The bifurcation diagram should have h as the horizontal axis and the critical points (call it c) as the vertical axis. The curve should be a parabola opening up to the left. The parabola should intersect the vertical axis at $c = 0$ and $c = 12$. The vertex is at $h = 3$.

7. (5 points) An object with mass 1 Kg is dropped from some initial height. Given that $g = 9.8m/s^2$, and assuming a force due to air resistance of $\frac{1}{2}v$, find the velocity function $v(t)$.

Solution: Force balance gives us $v' = 9.8 - \frac{1}{2}v$. Solving with $v(0) = 0$, we get $v(t) = 19.6(1 - e^{-(1/2)t})$.

8. Consider the initial value problem:

$$\begin{aligned}\frac{dy}{dt} &= f(y, t) \\ y(t_0) &= y_0.\end{aligned}$$

- (a) (1 point) What is the approximate magnitude of the error in Euler's method with a step-size of Δt ?

Solution: $e = O(\Delta t)$.

- (b) (2 points) Why is it not always reasonable to use a more accurate method like RK2 or RK4? (HINT: the answer depends on the nature of $y(t)$).

Solution: $y(t)$ must be sufficiently smooth to allow the use of RK2 or RK4. The constant in front of the error term depends on the higher derivatives of $y(t)$.

- (c) (2 points) Imagine you solve the ODE out to time t_1 with one step of Euler's method. What happens if you solve the ODE with a time-step $-\Delta t$ and the initial condition $y(t_1) = y_1$? Do you recover the same y_0 you began with? Justify your answer briefly.

Solution: You will not recover y_0 . Instead, the step backwards will incur an error of Δt as well. The result will be off from y_0 by a quantity that is $O(\Delta t)$.

9. (5 points) Mark as True or False. Justify *very* briefly.

- (a) Consider the IVP $y'(x) = f(x)$ with initial condition $y(0) = c$. If $f(c) = 0$, then $\lim_{x \rightarrow \infty} y(x) = c$.

Solution: T

- (b) Every solution to any differential equation can be obtained from the general solution if an initial condition is provided.

Solution: F

- (c) Linear first-order ODEs can have singular solutions in rare cases.

Solution: F

- (d) Autonomous ODEs have right hand sides that only involve the independent variable.

Solution: F

- (e) Let $\frac{dy}{dt} = 1 + y^2$. Then, the solution $y(t)$ will be valid for all t .

Solution: F

10. (5 points) (EXTRA CREDIT) Derive the fourth-order Runge Kutta method for the differential equation $\frac{dy}{dx} = f(x, y)$.

Hint: This is done by using Simpson's rule (given below) to approximate the integral $\int_{x_n}^{x_n+h} y'(x)dx$:

$$\int_a^b f(x)dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].$$