Lecture 25: Parallel Algorithms I

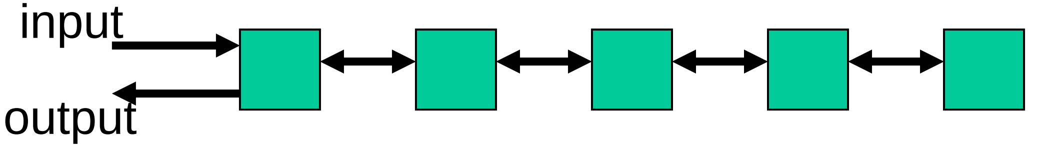
- Topics: sort and matrix algorithms

Processor Model

- High communication latencies → pursue coarse-grain parallelism (the focus of the course so far)
- For upcoming lectures, focus on fine-grain parallelism
- VLSI improvements → enough transistors to accommodate numerous processing units on a chip and (relatively) low communication latencies
- Consider a special-purpose processor with thousands of processing units, each with small-bit ALUs and limited register storage

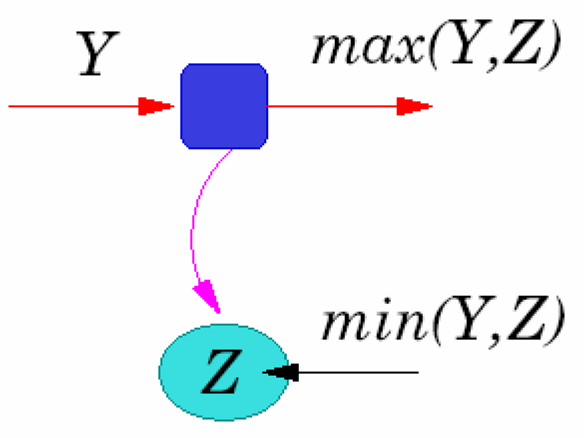
Sorting on a Linear Array

- Each processor has bidirectional links to its neighbors
- All processors share a single clock (asynchronous designs will require minor modifications)
- At each clock, processors receive inputs from neighbors, perform computations, generate output for neighbors, and update local storage

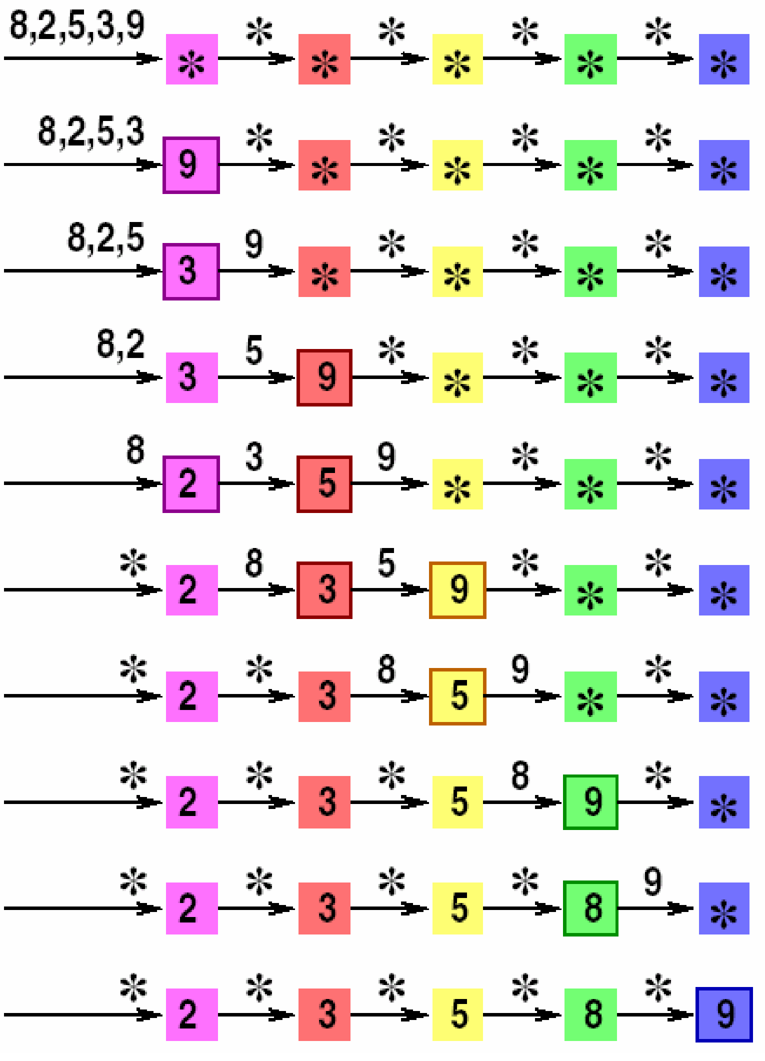


Control at Each Processor

- Each processor stores the minimum number it has seen
- Initial value in storage and on network is "$*$", which is bigger than any input and also means "no signal"
- On receiving number Y from left neighbor, the processor keeps the smaller of Y and current storage Z, and passes the larger to the right neighbor



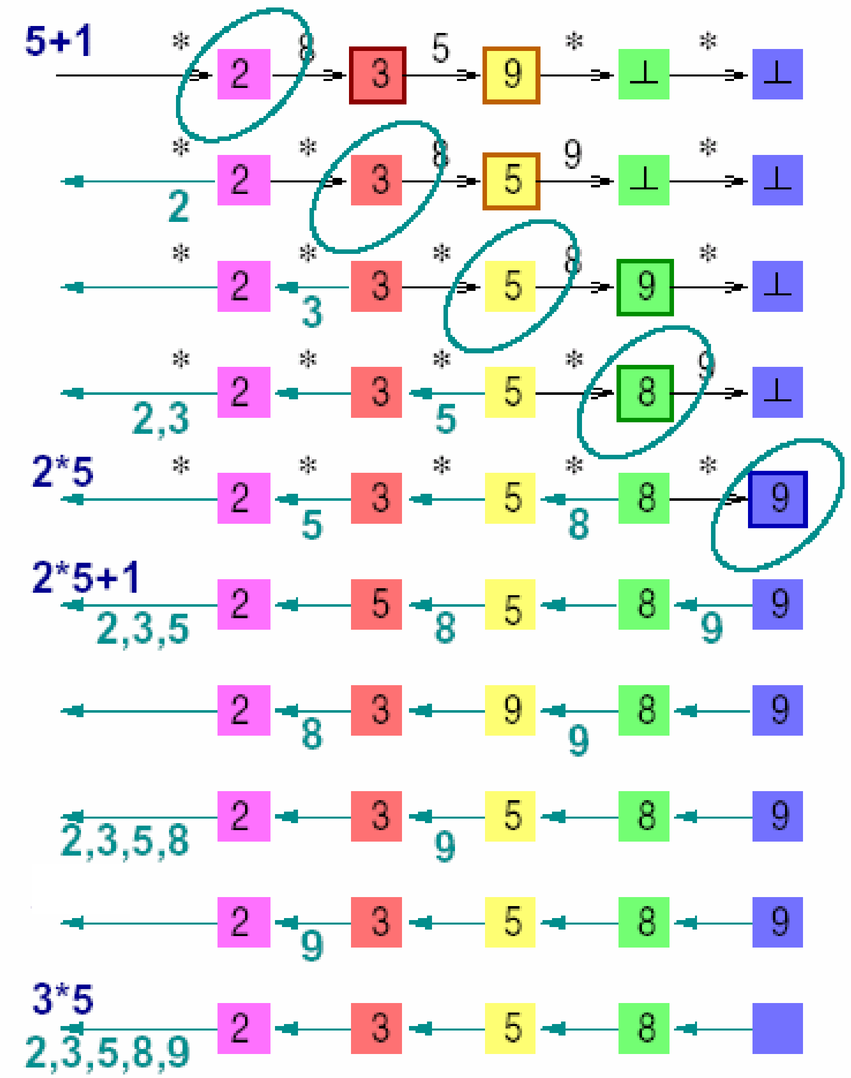
Sorting Example



Result Output

- The output process begins when a processor receives a non-$*$, followed by a "$*$"
- Each processor forwards its storage to its left neighbor and subsequent data it receives from right neighbors
- How many steps does it take to sort N numbers?
- What is the speedup and efficiency?

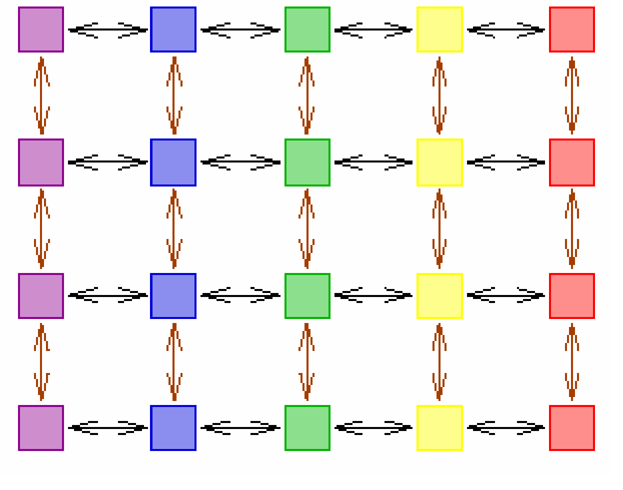
Output Example



Bit Model

- The bit model affords a more precise measure of complexity – we will now assume that each processor can only operate on a bit at a time

- To compare N k-bit words, you may now need an N x k 2-d array of bit processors



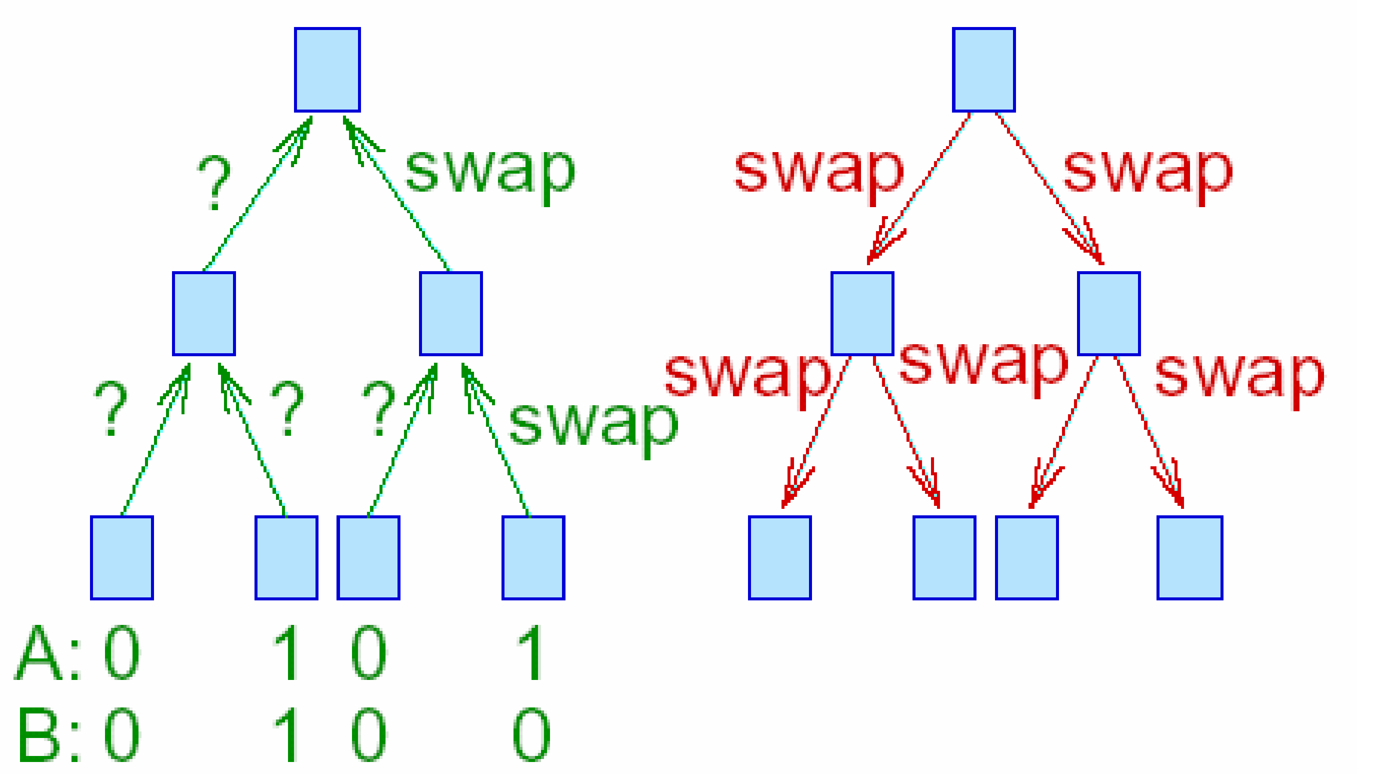
Comparison Strategies

- Strategy 1: Bits travel horizontally, keep/swap signals travel vertically – after at most 2k steps, each processor knows which number must be moved to the right – 2kN steps in the worst case

- Strategy 2: Use a tree to communicate information on which number is greater – after 2logk steps, each processor knows which number must be moved to the right – 2Nlogk steps

- Can we do better?

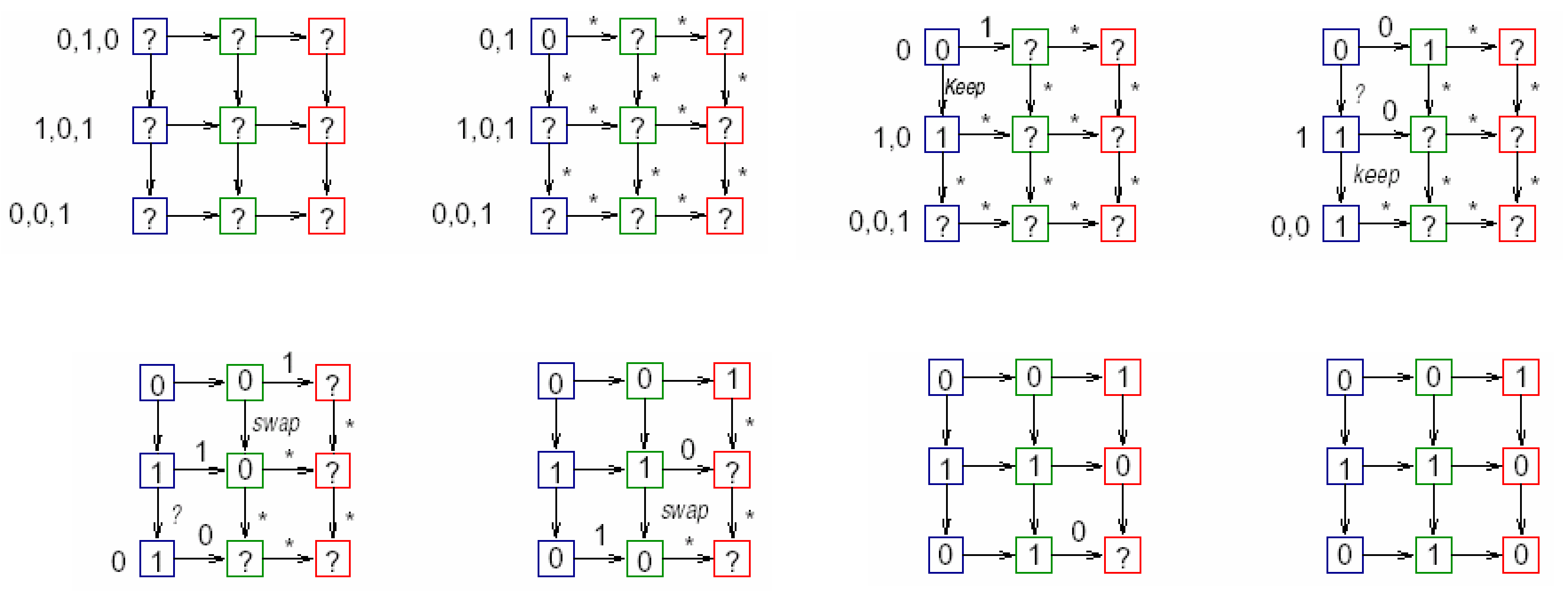
Strategy 2: Column of Trees



Pipelined Comparison

Input numbers:

3	4	2
0	1	0
1	0	1
1	0	0



Complexity

- How long does it take to sort N k-bit numbers?
  $(2N - 1) + (k - 1) + N$ (for output)

- (With a 2d array of processors) Can we do even better?

- How do we prove optimality?

Lower Bounds

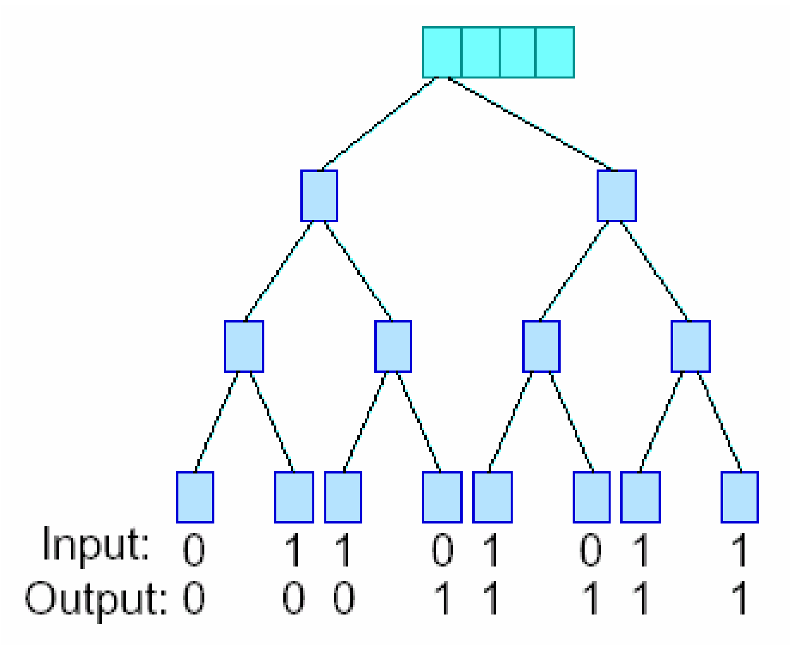
- Input/Output bandwidth: Nk bits are being input/output with k pins – requires $\Omega(N)$ time

- Diameter: the comparison at processor (1,1) influences the value of the bit stored at processor (N,k) – for example, N-1 numbers are 011..1 and the last number is either 00…0 or 10…0 – it takes at least N+k-2 steps for information to travel across the diameter

- Bisection width: if processors in one half require the results computed by the other half, the bisection bandwidth imposes a minimum completion time

Counter Example

- N 1-bit numbers that need to be sorted with a binary tree
- Since bisection bandwidth is 2 and each number may be in the wrong half, will any algorithm take at least N/2 steps?



Counting Algorithm

- It takes O(logN) time for each intermediate node to add the contents in the subtree and forward the result to the parent, one bit at a time
- After the root has computed the number of 1's, this number is communicated to the leaves – the leaves accordingly set their output to 0 or 1
- Each half only needs to know the number of 1's in the other half (logN-1 bits) – therefore, the algorithm takes Ω(logN) time
- Careful when estimating lower bounds!

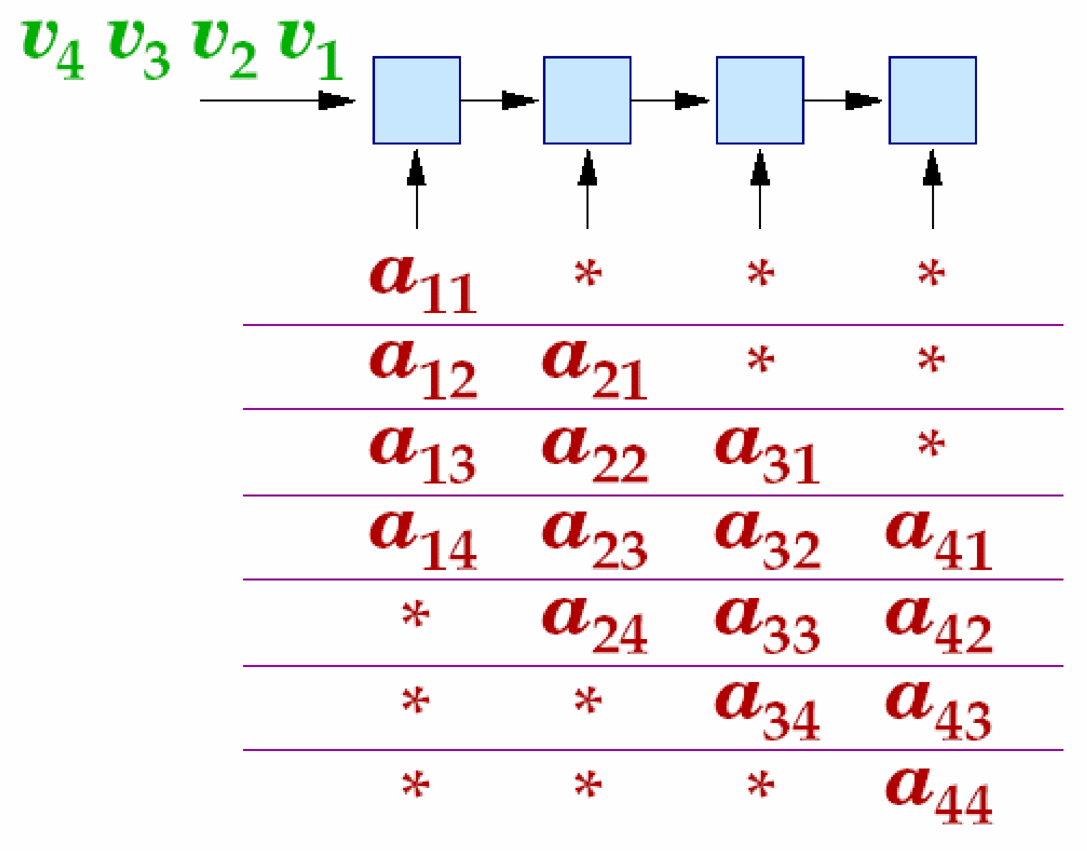
Matrix Algorithms

- Consider matrix-vector multiplication:

$$y_i = \Sigma_j \, a_{ij} x_j$$

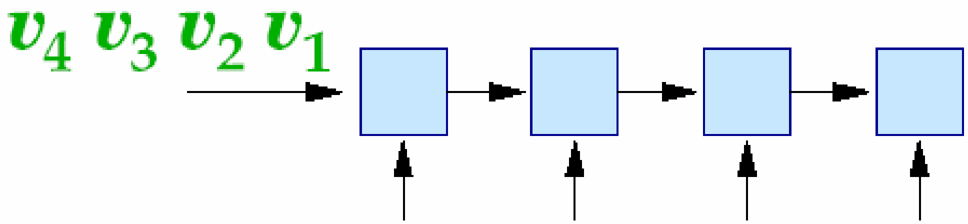
- The sequential algorithm takes $2N^2 - N$ operations
- With an N-cell linear array, can we implement matrix-vector multiplication in O(N) time?

Matrix Vector Multiplication



Number of steps = ?

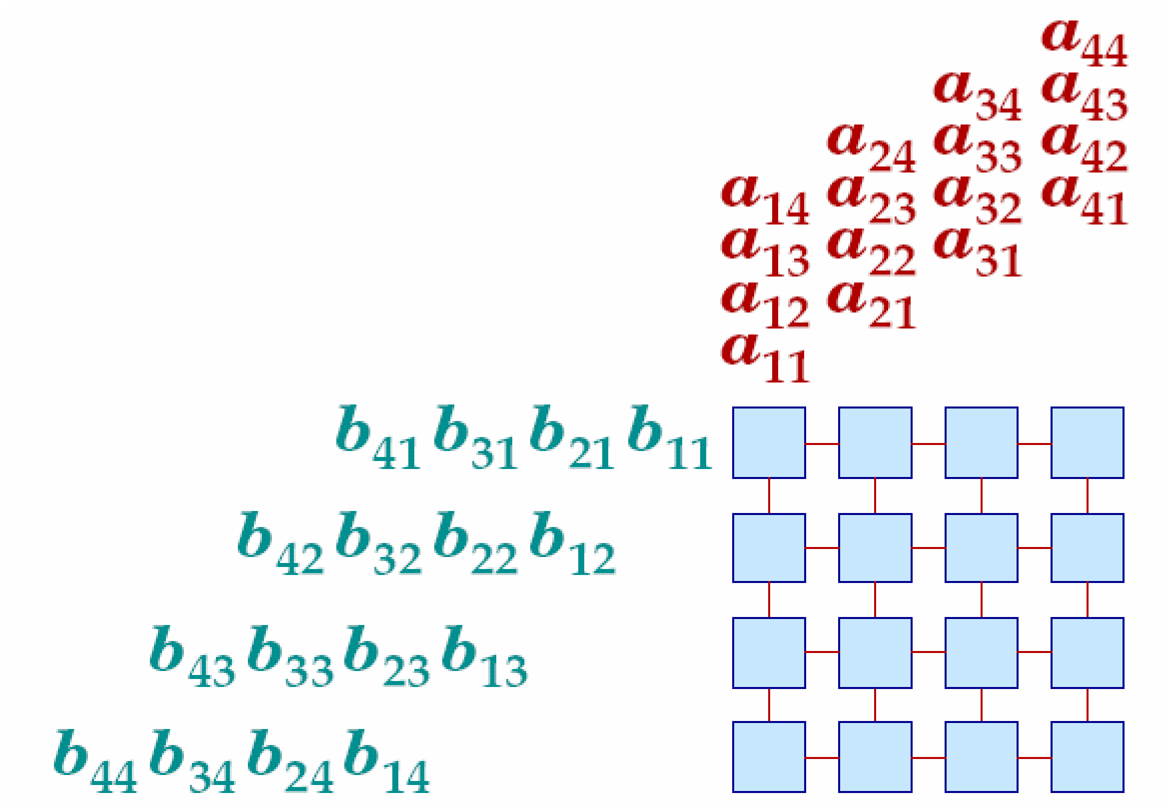
Matrix Vector Multiplication



a_{11}	*	*	*
a_{12}	a_{21}	*	*
a_{13}	a_{22}	a_{31}	*
a_{14}	a_{23}	a_{32}	a_{41}
*	a_{24}	a_{33}	a_{42}
*	*	a_{34}	a_{43}
*	*	*	a_{44}

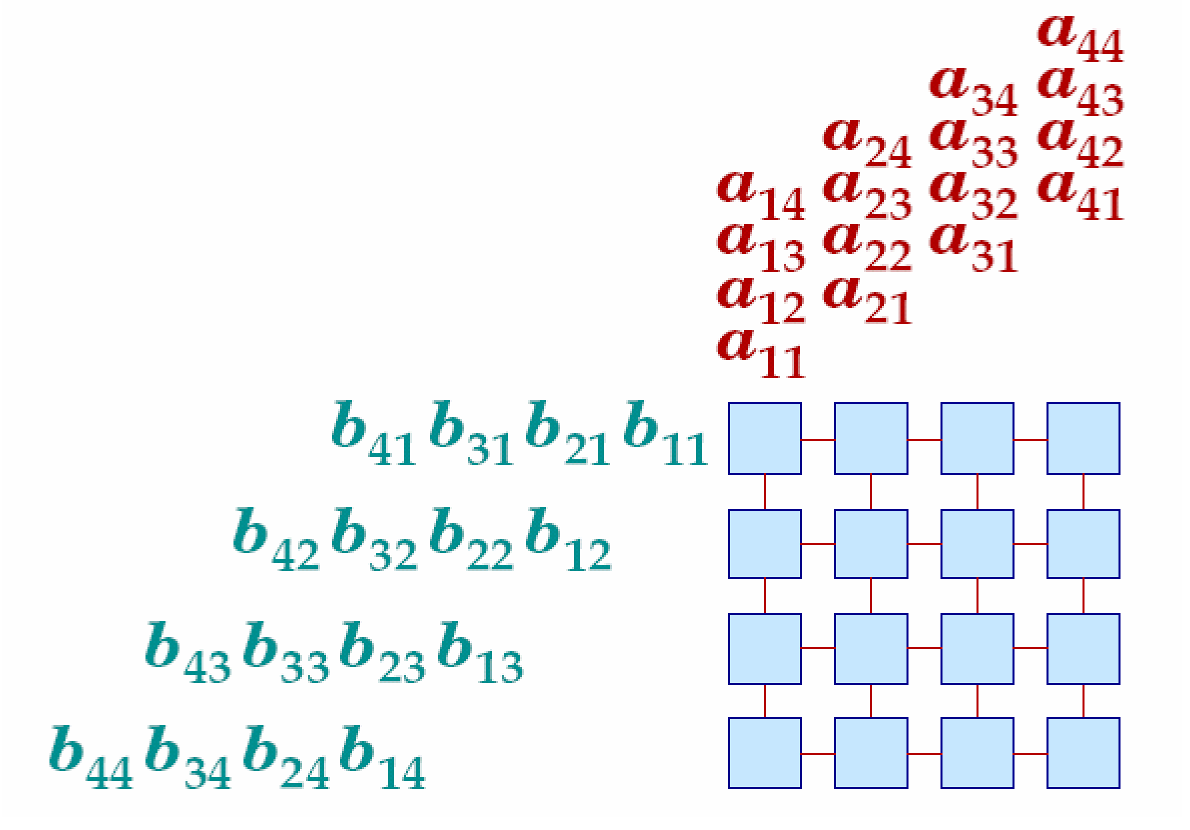
Number of steps = 2N – 1

Matrix-Matrix Multiplication



Number of time steps = ?

Matrix-Matrix Multiplication

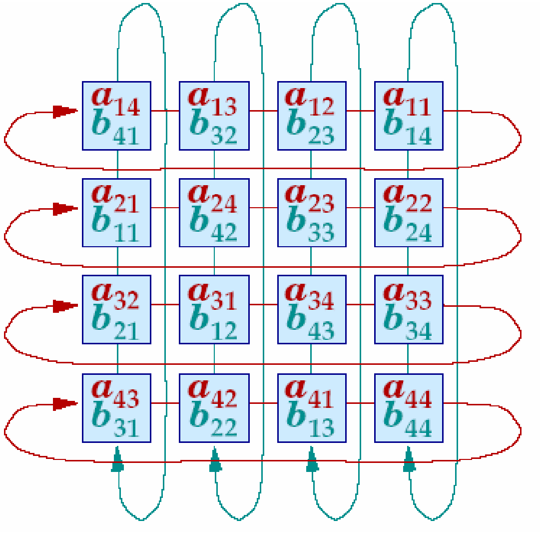


Number of time steps = 3N – 2

Complexity

- The algorithm implementations on the linear arrays have speedups that are linear in the number of processors – an efficiency of O(1)

- It is possible to improve these algorithms by a constant factor, for example, by inputting values directly to each processor in the first step and providing wraparound edges (N time steps)



Solving Systems of Equations

- Given an N x N lower triangular matrix A and an N-vector b, solve for x, where Ax = b (assume solution exists)

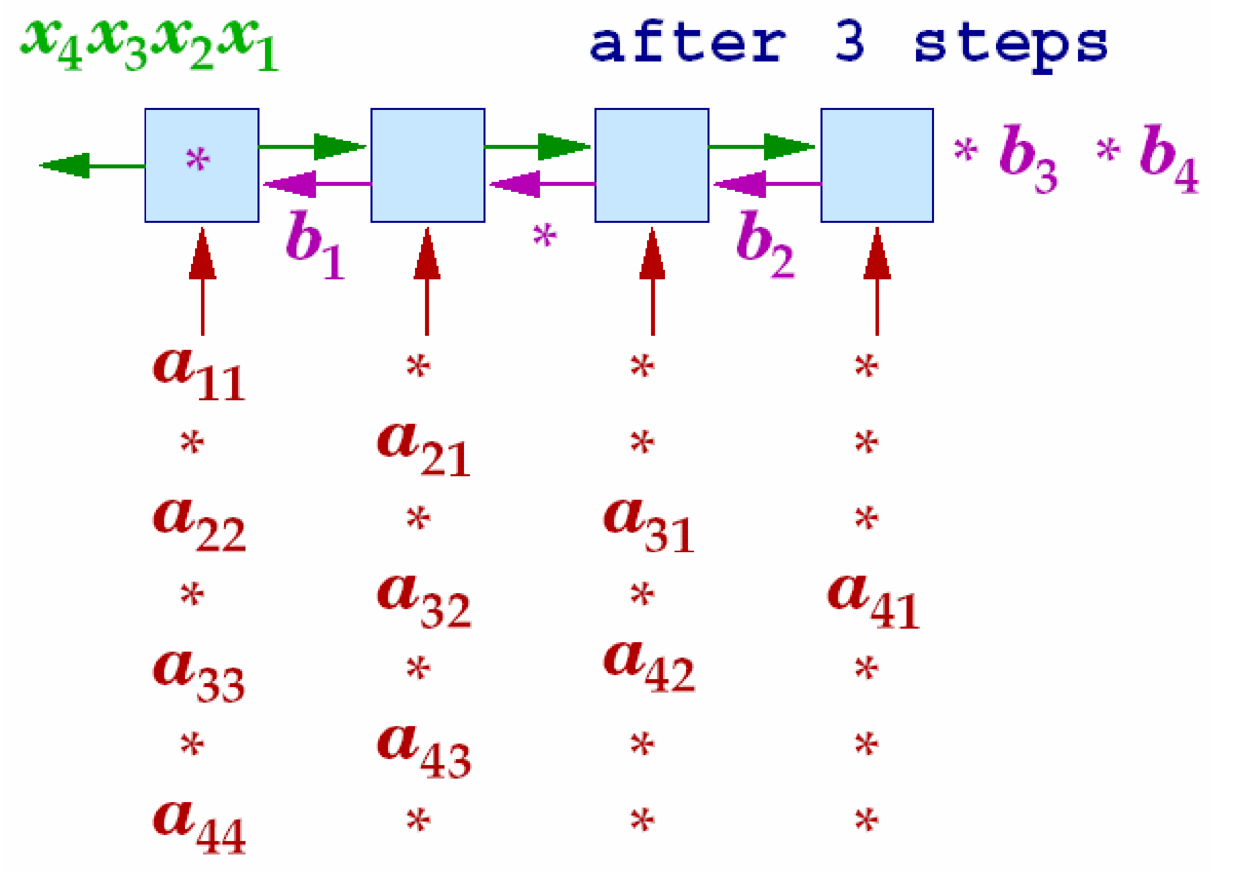
$a_{11}x_1 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$, and so on…

Define $t_1 =_{\text{def}} b_1$, $t_i =_{\text{def}} b_i - \sum_{j=1}^{i-1} a_{ij}x_j, 2 \leq i \leq N$. Then $x_i = t_i / a_{ii}$.

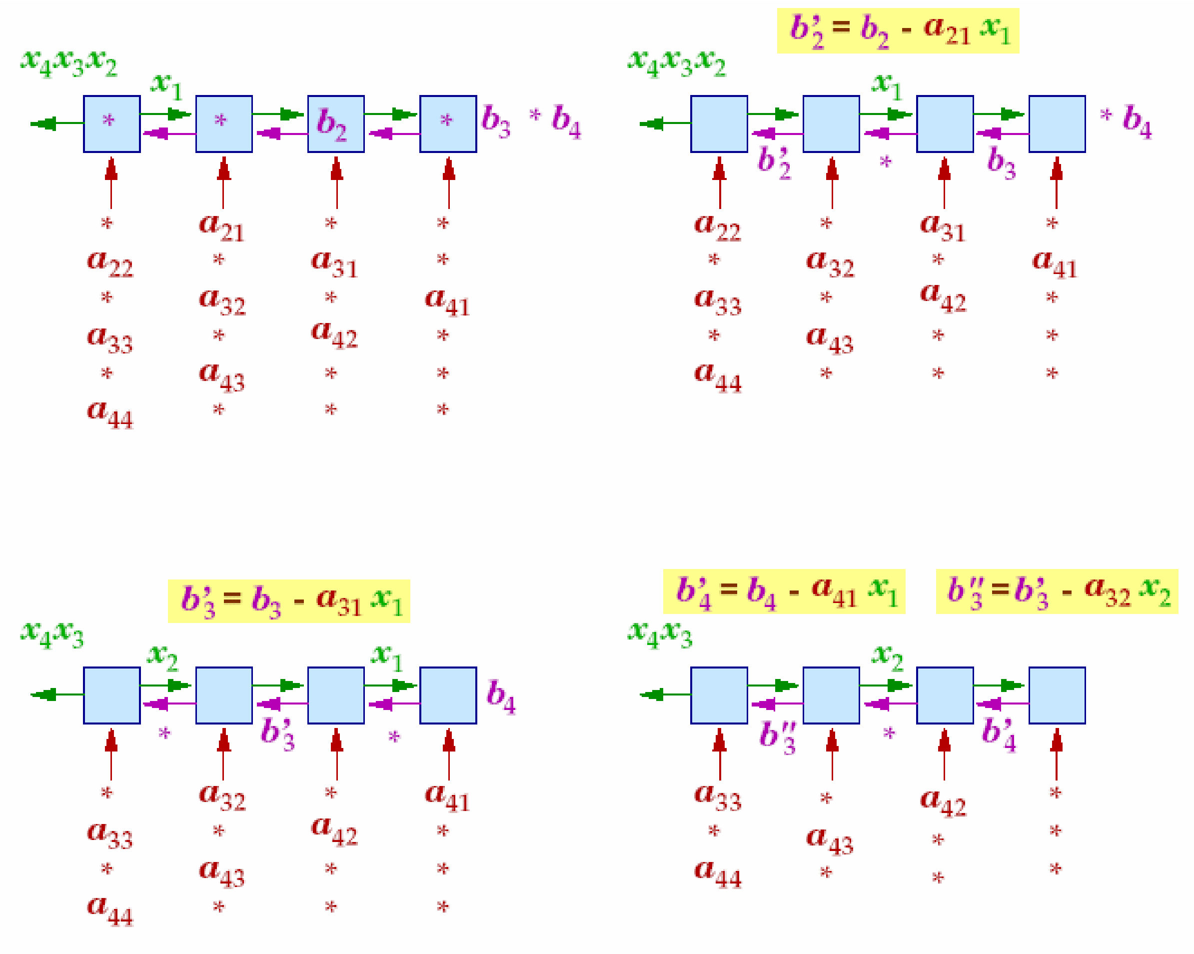
Equation Solver

Define $t_1 =_{\text{def}} b_1$, $t_i =_{\text{def}} b_i - \sum_{j=1}^{i-1} a_{ij}x_j, 2 \leq i \leq N$. Then $x_i = t_i/a_{ii}$.



Equation Solver Example

- When an *x, b,* and *a* meet at a cell, *ax* is subtracted from *b*
- When *b* and *a* meet at cell 1, *b* is divided by *a* to become *x*



Complexity

- Time steps = 2N – 1
- Speedup = O(N), efficiency = O(1)
- Note that half the processors are idle every time step – can improve efficiency by solving two interleaved equation systems simultaneously

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