Lecture 11: Floating Point, Digital Design

IEEE 754

Memory

bne — skipline

jal procA

label1:

skipline:

j label1

- Today's topics:

FP format

  - FP formats, arithmetic
  - Intro to Boolean functions

jal

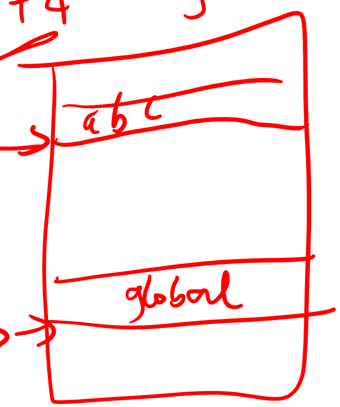
$ra = pc+4

.data

procA:

countervar word 75

$sp



$gp

jr $ra

la $t1, countervar

$23 \div 2 = 11$ rem 1

$11 \div 2 = 5$ rem 1

$5 \div 2 = 2$ rem 1

$2 \div 2 = 1$ rem 0

$1 \div 2 = 0$ rem 1

$0 \div 2 = 0$ rem 0

6×10^{-1}

$+ 4 \times 10^{-2}$

$\vdots$

0 0 0 0 1 0 1 1 1

$10111 = 23$?

750.9

$= 7.509$

$\times 10^{2}$

2^{-4}

$-4 \rightarrow 123$

0

0.647

$0.647 \times 2 = 1.294$

$0.294 \times 2 = 0.588$

$0.588 \times 2 = 1.176$

$0.176 \times 2 = 0.352$

$0.352 \times 2 = 0.704$

0.10100

23.647

$= 10111.10100 \ldots$

$1.011110100 \ldots \times 2^{4}$

$0.1 = 1 \times 10^{-1}$

$\rightarrow 0$

1000 0011 0 11110100 ....

Example 2

Final representation: $(-1)^S$ x (1 + Fraction) x $2^{(\text{Exponent} - \text{Bias})}$

- Represent 36.90625_{ten} in single-precision format

36 / 2 = 18 rem 0	0.90625 x 2 = 1.81250
18 / 2 = 9 rem 0	0.8125 x 2 = 1.6250
9 / 2 = 4 rem 1	0.625 x 2 = 1.250
4 / 2 = 2 rem 0	0.25 x 2 = 0.50
2 / 2 = 1 rem 0	0.5 x 2 = 1.00
1 / 2 = 0 rem 1	0.0 x 2 = 0.0

36 is 100100

0.90625 is 0.1110100…0

Example 2

Final representation: $(-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$

We've calculated that 36.90625_{ten} = 100100.1110100…0 in binary
Normalized form = 1.001001110100…0 x 2^5
(had to shift 5 places to get only one bit left of the point)

The sign bit is 0 (positive number)
The fraction field is 001001110100…0 (the 23 bits after the point)
The exponent field is 5 + 127 (have to add the bias) = 132,
which in binary is 10000100

The IEEE 754 format is 0 10000100 001001110100…..0
sign exponent 23 fraction bits

More Examples

$0.75 \times 2 = 1.50$
$0.5 \times 2 = 1.00$
$0.00 \times 2 = 0.00$

$0.75_{10} = 0.11000 \ldots_2$

Final representation: $(-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$

1.1×2^{-1}

- Represent -0.75_{ten} in single and double-precision formats

-1 true exp $\xrightarrow{+127}$ 126

Single: (1 + 8 + 23)

$\underbrace{1}_{\text{sign}} \quad \underbrace{}_{\text{exp}} \quad \underbrace{10000 \cdots 0}_{\text{frac}}$

Double: (1 + 11 + 52)

Remember:		
True exponent	+127 → ← -127	Exponent in register

- What decimal number is represented by the following single-precision number?

 1 $\underbrace{1000\ 0001}_{129}$ 01000…0000

$129 \xrightarrow{-127} +2$

$-\underline{1.0100} \times 2^2$
$= -101.00$
$= -5$

More Examples

Final representation: $(-1)^S \times (1 + \text{Fraction}) \times 2^{(\text{Exponent} - \text{Bias})}$

- Represent -0.75_{ten} in single and double-precision formats

Single: (1 + 8 + 23)

true exp -1 +1023 1022

1 $\underbrace{\text{0111 1110}}_{126}$ 1000…000

sign

Double: (1 + 11 + 52)

1 0111 1111 110 $\underbrace{\text{1000…000}}_{52 \text{ bit}}$

- What decimal number is represented by the following single-precision number?

 1 1000 0001 01000…0000

 -5.0

Value inf	2 special cases up top that use the reserved exponent field of 255	0 255 00…0
Value NAN		0 255 xx….x
Highest value ~2 x 2^{127}		0 254 11….1
Value 1		0 127 00…0
	Exponent field < 127, i.e., after subtracting bias, they are negative exponents, representing numbers < 1	
Smallest Norm ~2 x 2^{-126}		0 0..01 00…0
Largest Denorm ~1 x 2^{-126}	Special case with exponent field 0, used to represent denorms, that help us gradually approach 0	0 0..00 11…1
Smallest Denorm ~2^{-149}		0 0..00 00…1
Value 0		0 00..0 00…0

1

Same rules as above, but the sign bit is 1

Same magnitudes as above, but negative numbers

$\cancel{1}.000\ldots\ldots01$

$2^{-23} \times 2^{-126} = 2^{-149}$

1×2^{-127}

0

FP Addition

- Consider the following decimal example (can maintain only 4 decimal digits and 2 exponent digits)

shift exp left by 2

$9.999 \times 10^1 + 1.610 \times 10^{-1}$ +2

Convert to the larger exponent:

$9.999 \times 10^1 + 0.016 \times 10^1$

$$\begin{array}{r} 9.999 \\ +\ 0.016 \\ \hline 10.015 \end{array}$$

Add

10.015×10^1

~~Normalize~~

1.0015×10^2

Check for overflow/underflow

Round

1.002×10^2

Re-normalize

FP Addition

- Consider the following decimal example (can maintain only 4 decimal digits and 2 exponent digits)

$9.999 \times 10^{1} + 1.610 \times 10^{-1}$

Convert to the larger exponent:

$9.999 \times 10^{1} + 0.016 \times 10^{1}$

Add

10.015×10^{1}

Normalize

1.0015×10^{2}

Check for overflow/underflow

Round

1.002×10^{2}

Re-normalize

If we had more fraction bits, these errors would be minimized

FP Addition – Binary Example

- Consider the following binary example

$1.010 \times 2^1 \quad + \quad 1.100 \times 2^3$

Convert to the larger exponent:

$0.0101 \times 2^3 \quad + \quad 1.1000 \times 2^3$

Add

1.1101×2^3

Normalize

1.1101×2^3

Check for overflow/underflow

Round

Re-normalize

IEEE 754 format: 0 10000010 11010000000000000000000

FP Multiplication

- Similar steps:
  - Compute exponent (careful!)
  - Multiply significands (set the binary point correctly)
  - Normalize
  - Round (potentially re-normalize)
  - Assign sign

0 | | 1101...

adder

Multiplier

bias

$$\begin{array}{r} 1.1101 \\ \times 1.111001 \\ \hline 11.10\ldots \end{array}$$

$$\begin{array}{r} 3+127 \\ 4+127 \\ \hline 7+127 \end{array}$$

3.5×10^{3}

$\times\ 2.3 \times 10^{4}$

$$\begin{array}{r} 3.5 \\ \times\ 2.3 \\ \hline 8.\ldots \end{array} \times 10^{7}$$

MIPS Instructions

.d → double prec

INT

Co-proc FP

- The usual add.s, add.d, sub, mul, div

- Comparison instructions: c.eq.s, c.neq.s, c.lt.s….
  These comparisons set an internal bit in hardware that is then inspected by branch instructions: bc1t, bc1f

64b 2 32b regs

- Separate register file \$f0 - \$f31 : a double-precision value is stored in (say) \$f4-\$f5 and is referred to by \$f4

- Load/store instructions (lwc1, swc1) must still use integer registers for address computation

Code Example

```
float  f2c (float fahr)
{
    return ((5.0/9.0) * (fahr – 32.0));
}
```

.data
const5   float   5.0

(argument fahr is stored in $f12)

```
lwc1   $f16, const5
lwc1   $f18, const9
div.s  $f16, $f16, $f18
lwc1   $f18, const32
sub.s  $f18, $f12, $f18
mul.s  $f0, $f16, $f18
jr     $ra
```

Fixed Point

INT ADD 1cyc | INT MUL 4cyc | FP ADD 5cyc | FP MUL 7cyc

- FP operations are much slower than integer ops

0.00332..

fixed point

Google TPU

- Fixed point arithmetic uses integers, but assumes that every number is multiplied by the same factor

- Example: with a factor of 1/1000, the fixed-point representations for 1.46, 1.7198, and 5624 are respectively $\frac{1}{1000}$ 1460, 1720, and 5624000

- More programming effort and possibly lower precision for higher performance

Subword Parallelism

carry

| B | G | R | 32 bit INTS

+

| B | G | R |

- ALUs are typically designed to perform 64-bit or 128-bit arithmetic

ADD

SIMD
single instr mult data

img

RGB

- Some data types are much smaller, e.g., bytes for pixel RGB values, half-words for audio samples

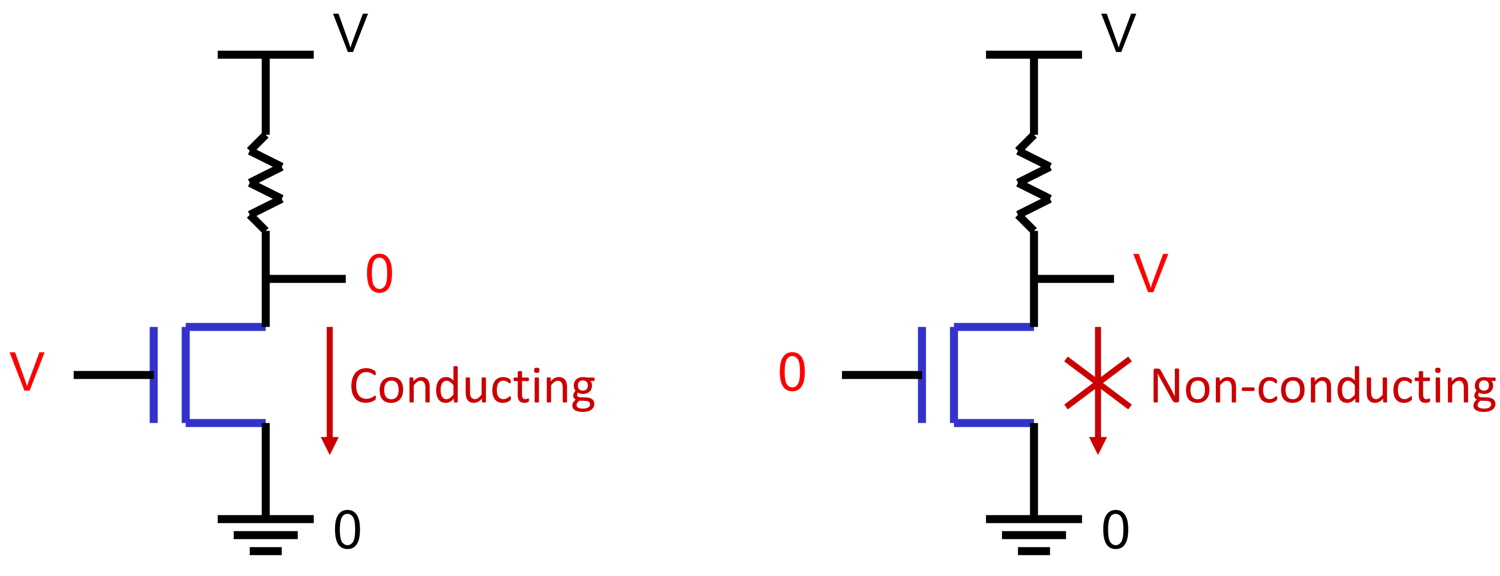
- Partitioning the carry-chains within the ALU can convert the 64-bit adder into 4 16-bit adders or 8 8-bit adders

- A single load can fetch multiple values, and a single add instruction can perform multiple parallel additions, referred to as subword parallelism

Digital Design Basics

- Two voltage levels – high and low (1 and 0, true and false)
  Hence, the use of binary arithmetic/logic in all computers

- A transistor is a 3-terminal device that acts as a switch



Logic Blocks

- A logic block has a number of binary inputs and produces a number of binary outputs – the simplest logic block is composed of a few transistors
- A logic block is termed *combinational* if the output is only a function of the inputs
- A logic block is termed *sequential* if the block has some internal memory (state) that also influences the output
- A basic logic block is termed a *gate* (AND, OR, NOT, etc.)

We will only deal with combinational circuits today

Truth Table

- A truth table defines the outputs of a logic block for each set of inputs

- Consider a block with 3 inputs A, B, C and an output E that is true only if *exactly* 2 inputs are true

A	B	C	E

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- A truth table defines the outputs of a logic block for each set of inputs

- Consider a block with 3 inputs A, B, C and an output E that is true only if *exactly* 2 inputs are true

A	B	C	E
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Can be compressed by only representing cases that have an output of 1

Boolean Algebra

- Equations involving two values and three primary operators:
  - OR : symbol + , X = A + B ➔ X is true if at least one of A or B is true
  - AND : symbol **.** , X = A **.** B ➔ X is true if both A and B are true
  - NOT : symbol $\bar{\ }$, X = $\bar{A}$ ➔ X is the inverted value of A

Boolean Algebra Rules

- Identity law : $A + 0 = A$; $A . 1 = A$
- Zero and One laws : $A + 1 = 1$; $A . 0 = 0$
- Inverse laws : $A . \overline{A} = 0$; $A + \overline{A} = 1$
- Commutative laws : $A + B = B + A$; $A . B = B . A$
- Associative laws : $A + (B + C) = (A + B) + C$
  $A . (B . C) = (A . B) . C$
- Distributive laws : $A . (B + C) = (A . B) + (A . C)$
  $A + (B . C) = (A + B) . (A + C)$

DeMorgan's Laws

- $\overline{A + B} = \overline{A} \, . \, \overline{B}$

- $\overline{A \, . \, B} = \overline{A} + \overline{B}$

- Confirm that these are indeed true