Lecture 12: Hardware for Arithmetic

- Today's topics:
  - Logic for common operations
  - Designing an ALU

Truth Table

- A truth table defines the outputs of a logic block for each set of inputs
- Consider a block with 3 inputs A, B, C and an output E that is true only if *exactly* 2 inputs are true

A	B	C	E
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

Can be compressed by only representing cases that have an output of 1

Boolean Algebra

- Equations involving two values and three primary operators:
  - OR : symbol + , X = A + B ➔ X is true if at least one of A or B is true
  - AND : symbol **.** , X = A **.** B ➔ X is true if both A and B are true
  - NOT : symbol $\overline{}$, X = $\overline{A}$ ➔ X is the inverted value of A

Boolean Algebra Rules

- Identity law : $A + 0 = A$; $A . 1 = A$
- Zero and One laws : $A + 1 = 1$; $A . 0 = 0$
- Inverse laws : $A . \overline{A} = 0$; $A + \overline{A} = 1$
- Commutative laws : $A + B = B + A$; $A . B = B . A$
- Associative laws : $A + (B + C) = (A + B) + C$
  $A . (B . C) = (A . B) . C$
- Distributive laws : $A . (B + C) = (A . B) + (A . C)$
  $A + (B . C) = (A + B) . (A + C)$

DeMorgan's Laws

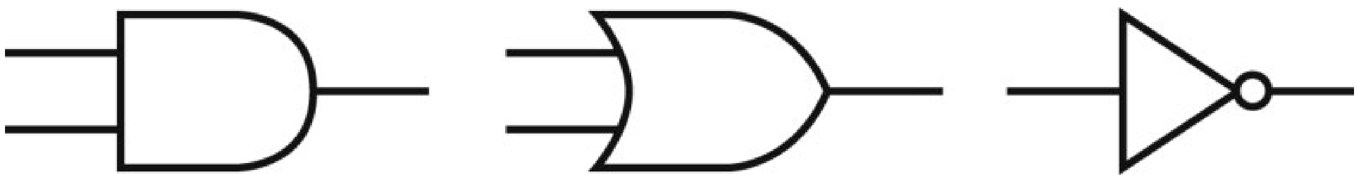
- $\overline{A + B} = \overline{A} \cdot \overline{B}$

- $\overline{A \cdot B} = \overline{A} + \overline{B}$

- Confirm that these are indeed true

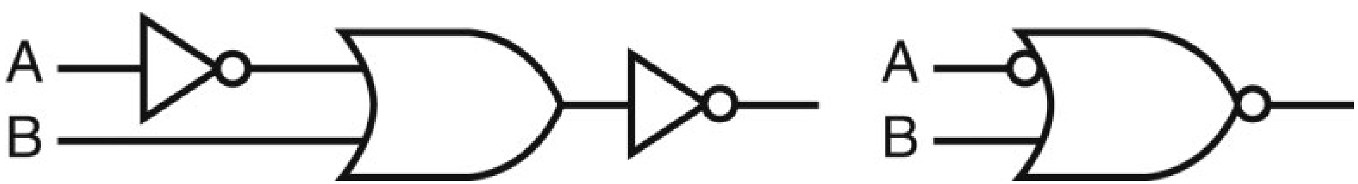
Pictorial Representations

AND OR NOT



Source: H&P textbook

What logic function is this?



Source: H&P textbook

Boolean Equation

- Consider the logic block that has an output E that is true only if exactly two of the three inputs A, B, C are true

  Multiple correct equations:

  Two must be true, but all three cannot be true:

  $E = ((A \,.\, B) + (B \,.\, C) + (A \,.\, C)) \,.\, (\overline{A \,.\, B \,.\, C})$

  Identify the three cases where it is true:

  $E = (A \,.\, B \,.\, \overline{C}) + (A \,.\, C \,.\, \overline{B}) + (C \,.\, B \,.\, \overline{A})$

Sum of Products

- Can represent any logic block with the AND, OR, NOT operators
  - Draw the truth table
  - For each true output, represent the corresponding inputs as a product
  - The final equation is a sum of these products

A	B	C	E
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

$$(A \,.\, B \,.\, \overline{C}) + (A \,.\, C \,.\, \overline{B}) + (C \,.\, B \,.\, \overline{A})$$

- Can also use "product of sums"
- Any equation can be implemented with an array of ANDs, followed by an array of ORs

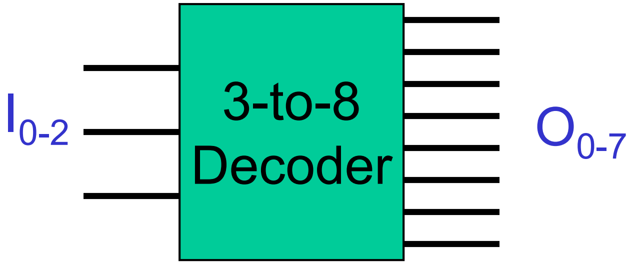
NAND and NOR

- NAND : NOT of AND : A nand B $= \overline{A . B}$
- NOR : NOT of OR : A nor B $= \overline{A + B}$
- NAND and NOR are *universal gates*, i.e., they can be used to construct any complex logical function

Common Logic Blocks – Decoder

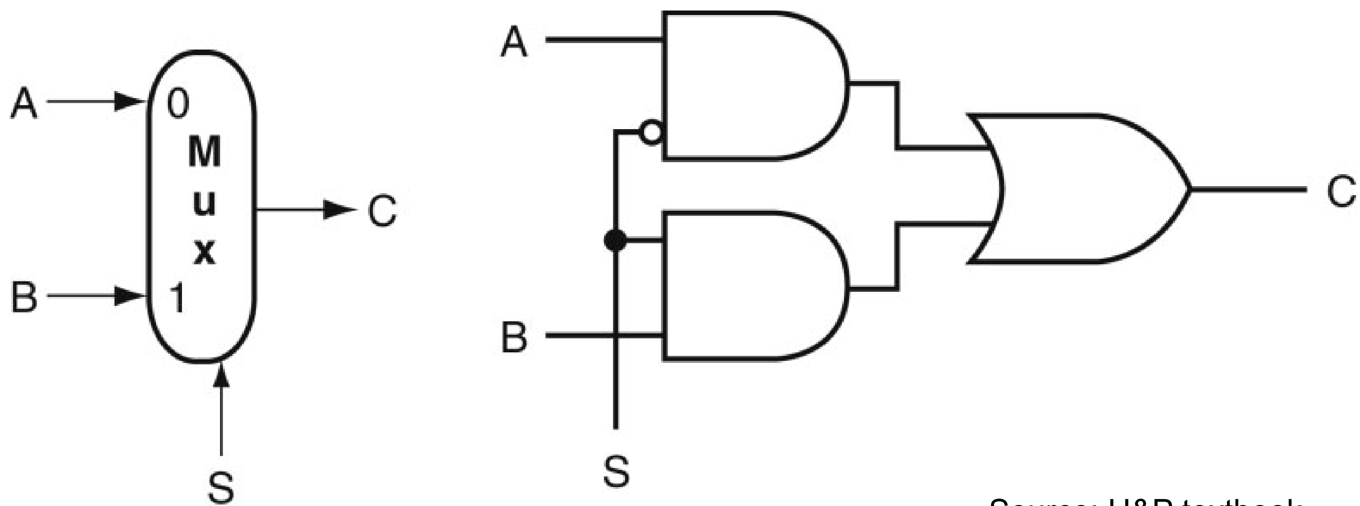
Takes in N inputs and activates one of 2^N outputs

I_0	I_1	I_2	O_0	O_1	O_2	O_3	O_4	O_5	O_6	O_7
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1



Common Logic Blocks – Multiplexor

- Multiplexor or selector: one of N inputs is reflected on the output depending on the value of the $\log_2 N$ selector bits



Source: H&P textbook

2-input mux

Adder Algorithm

	1	0	0	1
	0	1	0	1
Sum	1	1	1	0
Carry	0	0	0	1

Truth Table for the above operations:

A	B	Cin	Sum	Cout
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Adder Algorithm

	1	0	0	1
	0	1	0	1
Sum	1	1	1	0
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Truth Table for the above operations:

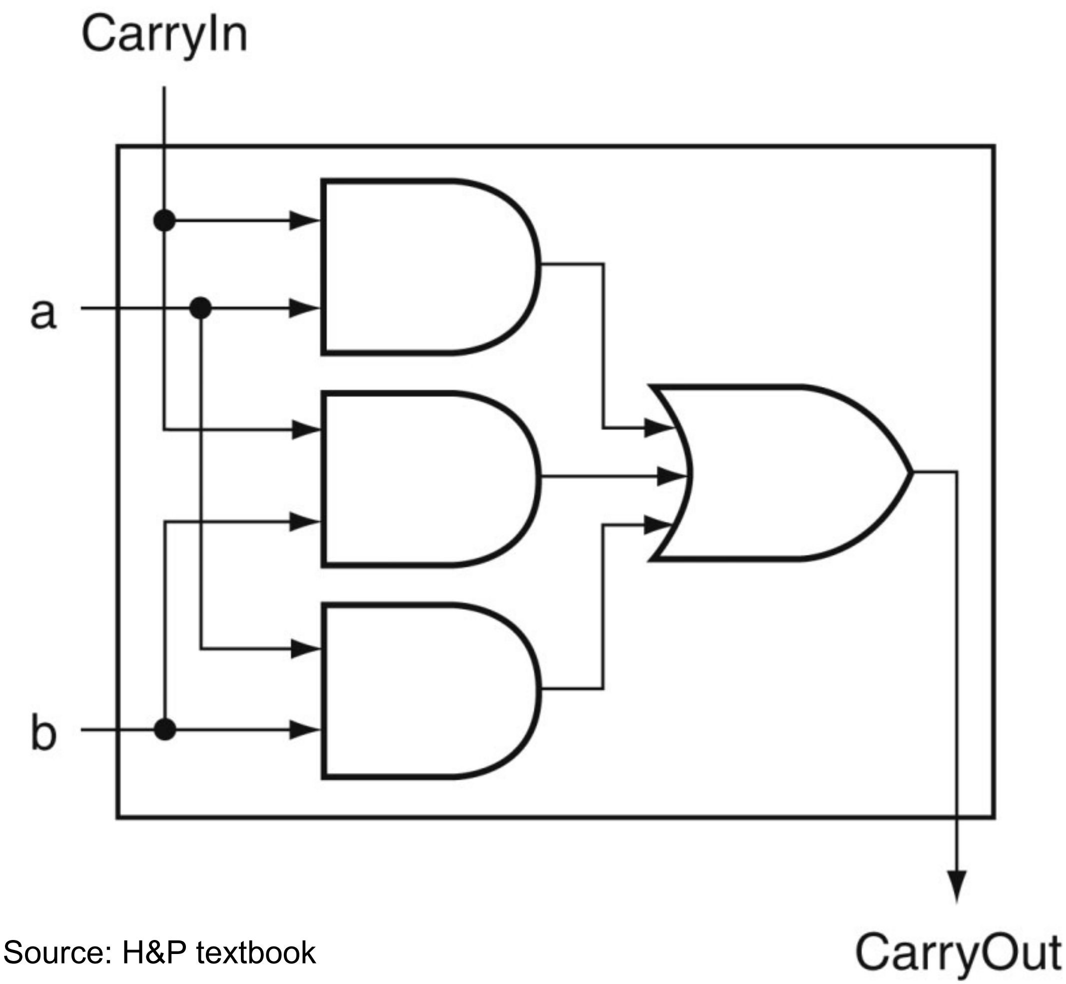
A	B	Cin	Sum	Cout
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Equations:

$$\begin{aligned} \text{Sum} = {} & \text{Cin} \,.\, \overline{A} \,.\, \overline{B} \,+ \\ & B \,.\, \overline{\text{Cin}} \,.\, \overline{A} \,+ \\ & A \,.\, \overline{\text{Cin}} \,.\, \overline{B} \,+ \\ & A \,.\, B \,.\, \text{Cin} \end{aligned}$$

$$\begin{aligned} \text{Cout} = {} & A \,.\, B \,.\, \text{Cin} \,+ \\ & A \,.\, B \,.\, \overline{\text{Cin}} \,+ \\ & A \,.\, \text{Cin} \,.\, \overline{B} \,+ \\ & B \,.\, \text{Cin} \,.\, \overline{A} \\ = {} & A \,.\, B \,+ \\ & A \,.\, \text{Cin} \,+ \\ & B \,.\, \text{Cin} \end{aligned}$$

Carry Out Logic



Source: H&P textbook

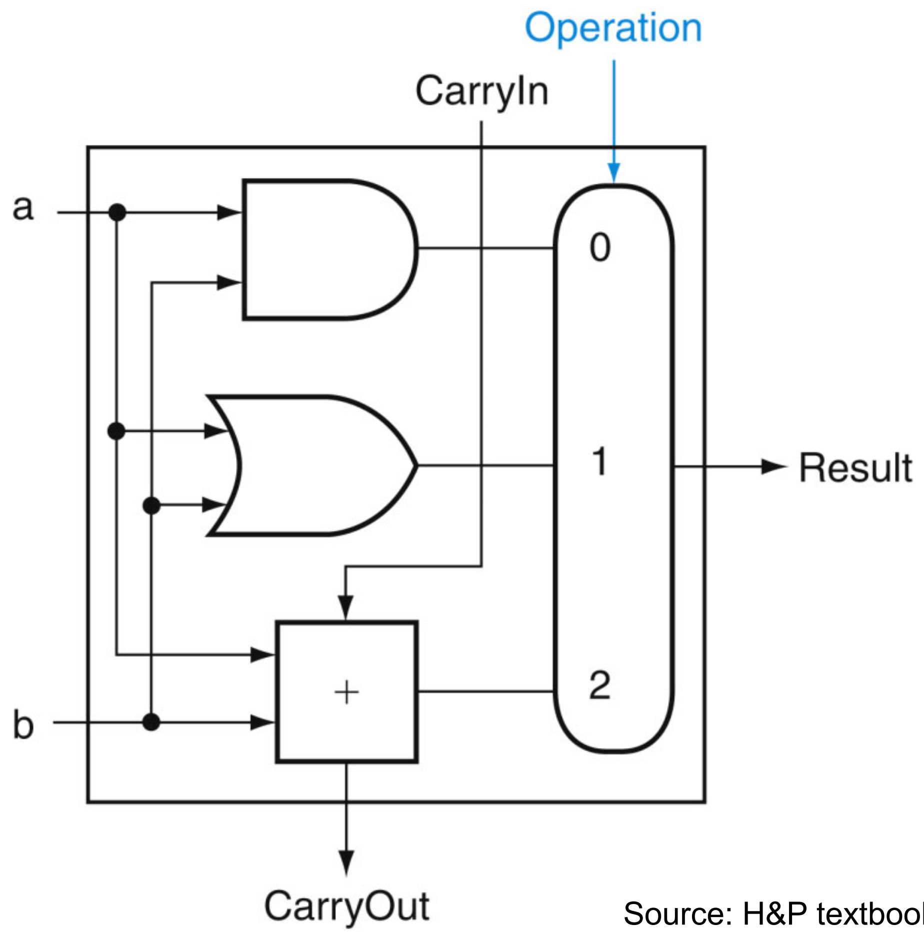
Equations:

$$\begin{aligned} Sum = {} & Cin \,.\, \overline{A} \,.\, \overline{B} \;+ \\ & B \,.\, \overline{Cin} \,.\, \overline{A} \;+ \\ & A \,.\, \overline{Cin} \,.\, \overline{B} \;+ \\ & A \,.\, B \,.\, Cin \end{aligned}$$

$$\begin{aligned} Cout = {} & A \,.\, B \,.\, Cin \;+ \\ & A \,.\, B \,.\, \overline{Cin} \;+ \\ & A \,.\, Cin \,.\, \overline{B} \;+ \\ & B \,.\, Cin \,.\, \overline{A} \\ = {} & A \,.\, B \;+ \\ & A \,.\, Cin \;+ \\ & B \,.\, Cin \end{aligned}$$

1-Bit ALU with Add, Or, And

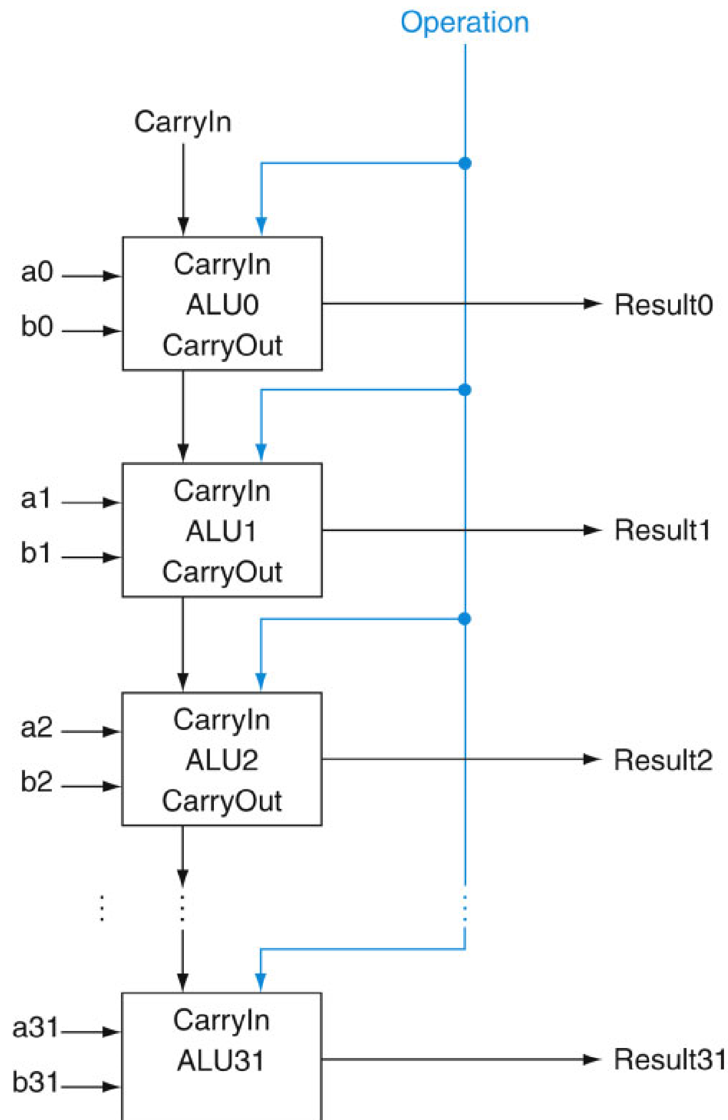
- Multiplexor selects between Add, Or, And operations



Source: H&P textbook

32-bit Ripple Carry Adder

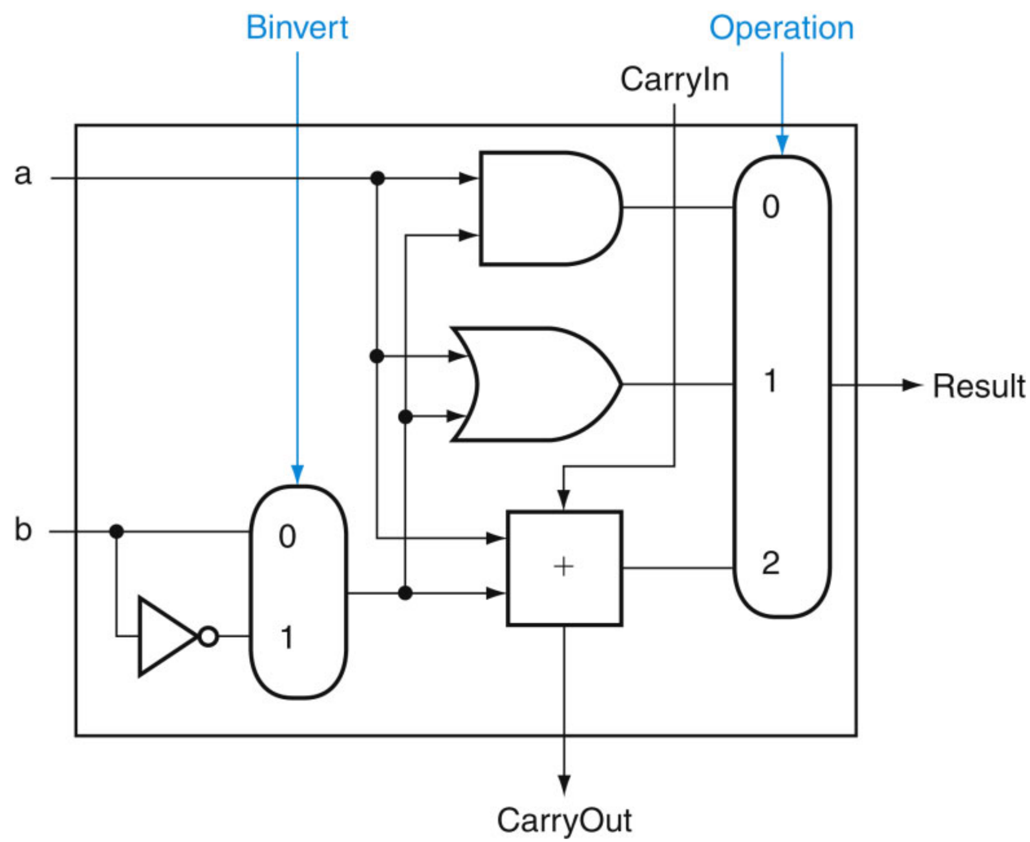
1-bit ALUs are connected "in series" with the carry-out of 1 box going into the carry-in of the next box



Source: H&P textbook

Incorporating Subtraction

Must invert bits of B and add a 1
- Include an inverter
- CarryIn for the first bit is 1
- The CarryIn signal (for the first bit) can be the same as the Binvert signal



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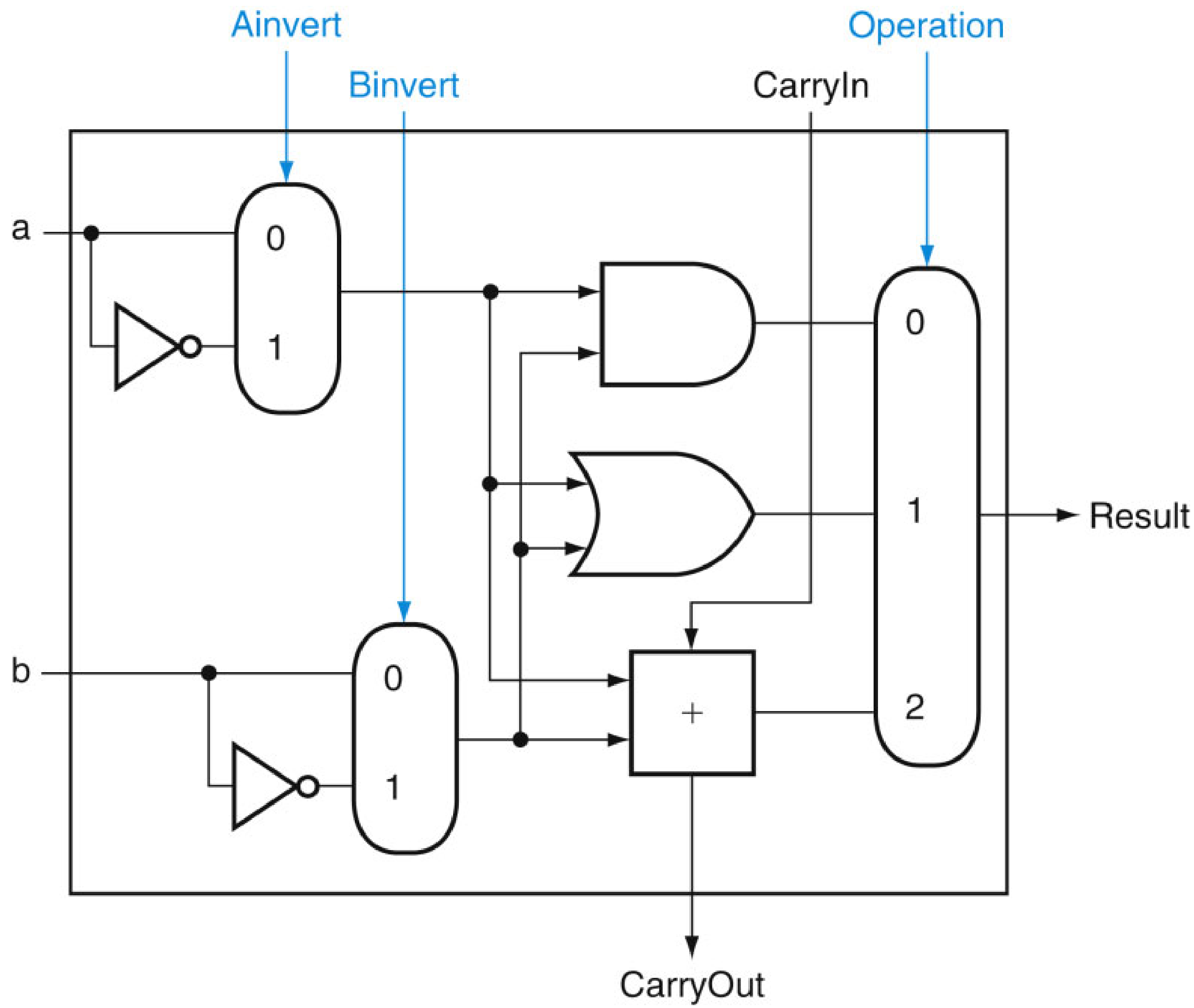
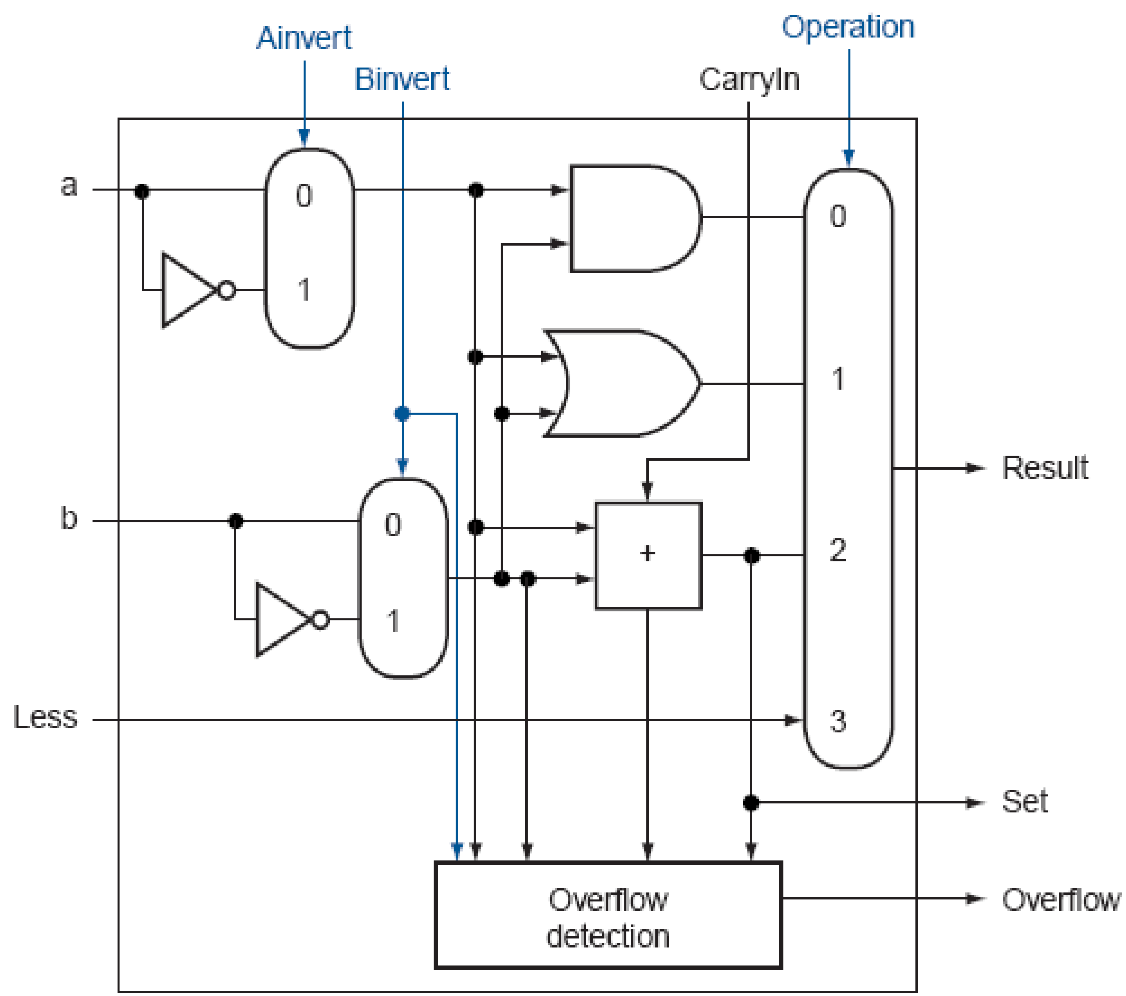
Incorporating NOR and NAND



Source: H&P textbook

Incorporating slt

- Perform a – b and check the sign
- New signal (Less) that is zero for ALU boxes 1-31
- The 31st box has a unit to detect overflow and sign – the sign bit serves as the Less signal for the 0th box



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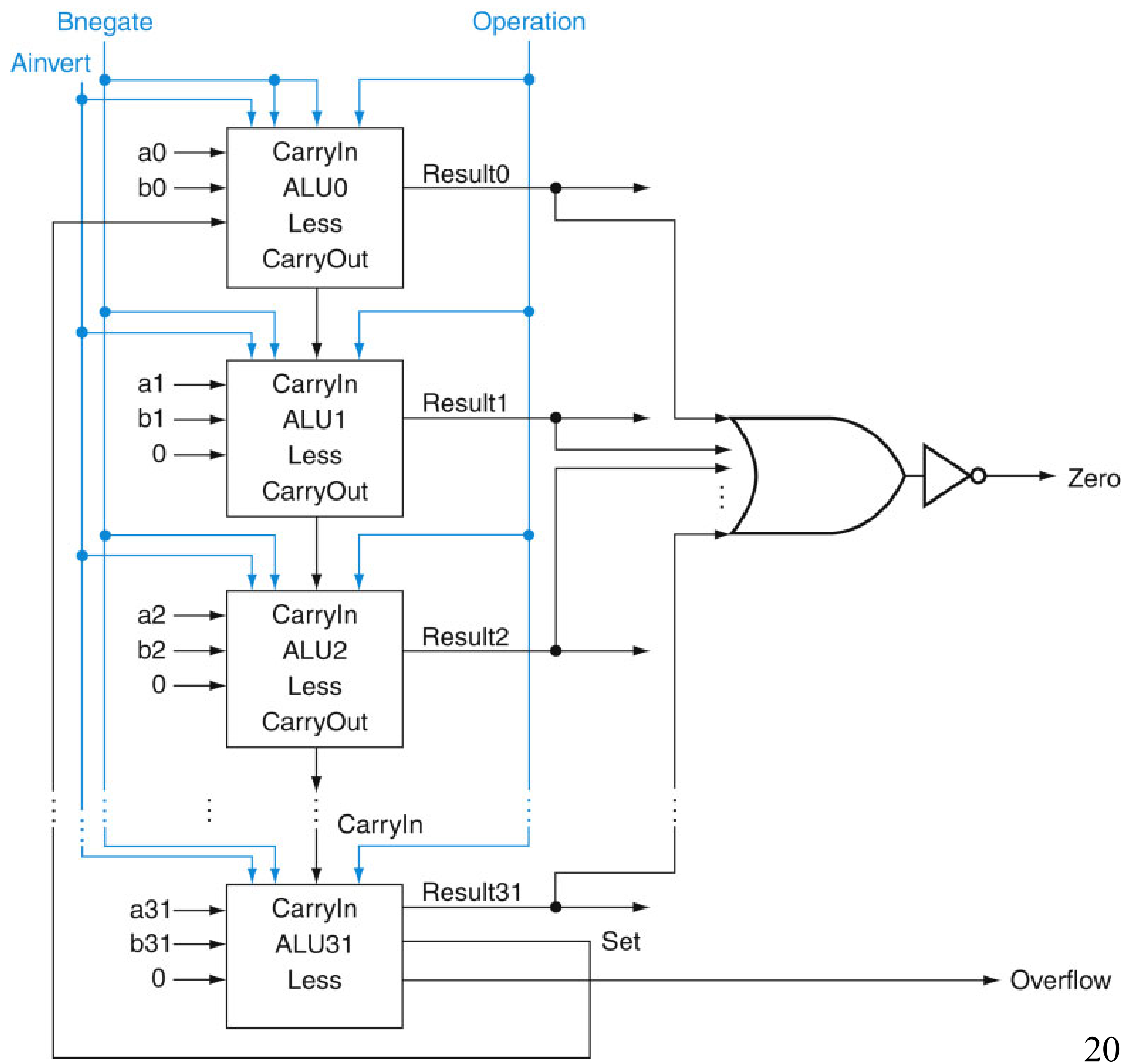
Incorporating beq

- Perform a – b and confirm that the result is all zero's



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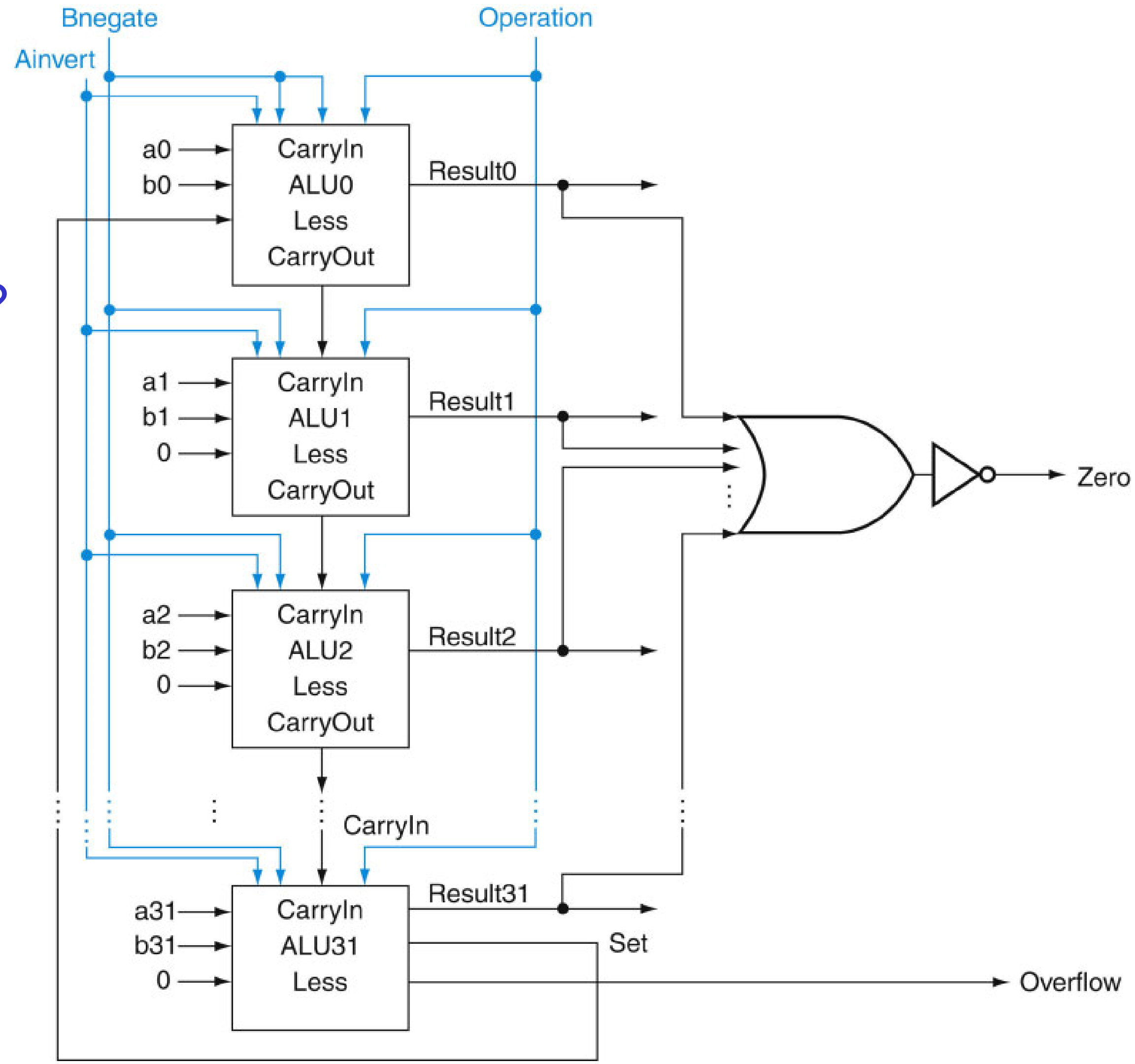
Control Lines

What are the values of the control lines and what operations do they correspond to?



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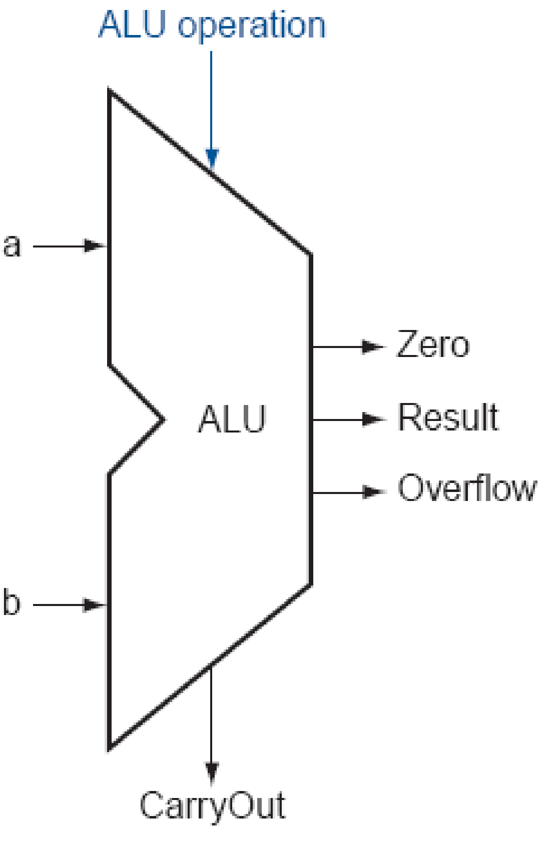
Control Lines

What are the values of the control lines and what operations do they correspond to?

	Ai	Bn	Op
AND	0	0	00
OR	0	0	01
Add	0	0	10
Sub	0	1	10
SLT	0	1	11
NOR	1	1	00



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