Maximum-Likelihood Estimation With Newton-Raphson Iteration

Vincent Kang Fu

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Background

- Newton-Raphson iteration for maximum-likelihood estimation is implemented in many professional statistical packages

- Can I write my own version in C that is faster than these highly optimized statistical packages?

Maximum Likelihood Estimation

- Prominent method for estimating statistical models
- Choose model coefficients to maximize the joint likelihood of the observed outcomes
- Statistical estimation becomes an optimization problem
- Use Newton-Raphson iteration for optimization

Newton-Raphson Iteration

- General method for optimizing a function
- Set the first derivative to zero and apply Newton's method to solve
- Iterative procedure:
  - β_0 = starting value
  - for $i = 0, 1, \cdots, \infty$ until convergence do

$$\beta_{i+1} = \beta_i - \frac{\partial^2 l}{\partial \beta^2}^{-1} \frac{\partial l}{\partial \beta}$$

Modeling Married Women's Labor Force Participation

- *N* = 1,984,591 married women from the 1990 US Census
- Model: Binary logit model

p_i = labor force participation probability

$$\log \frac{p_i}{(1-p_i)} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots$$

Data Arrays

- Y is an N x 1 vector where y_i = 1 for women in the labor force, 0 otherwise
- X is an N x K array of covariates including race, education, age, other income in the household, household size, presence of young children

Maximum Lilkelihood Estimation

- The *likelihood L* is the joint probability of the observed outcomes
- Find coefficients β to maximize L

$$L(\beta)=\prod_{y_i=1} p_i \prod_{y_i=0} (1-p_i), \quad p_i=\frac{\exp(\beta' x_i)}{1+\exp(\beta' x_i)}$$

- Easier to maximize l, the log likelihood
- Need the first and second derivatives for Newton-Raphson iteration

First Derivative

```
for (i = 0; i < N; i++)
        for (k = 0; k < REGRESSORS; k++)
                result[k] += (y[i] - p[i])*x[i][k];
```

$$\frac{\partial l}{\partial \beta} = \begin{bmatrix} \sum (y_i - p_i) x_{0i} \\ \sum (y_i - p_i) x_{1i} \\ \vdots \\ \sum (y_i - p_i) x_{Ki} \end{bmatrix}$$

- Calculating elements in parallel sacrifices locality in access to *X* array

Second Derivative

$$\frac{\partial^2 l}{\partial \beta^2} = -\begin{bmatrix} \sum p_i(1-p_i)x_{0i}^2 & \sum p_i(1-p_i)x_{0i}x_{1i} & \cdots & \sum p_i(1-p_i)x_{0i}x_{Ki} \\ \sum p_i(1-p_i)x_{1i}x_{0i} & \sum p_i(1-p_i)x_{1i}^2 & \cdots & \sum p_i(1-p_i)x_{1i}x_{Ki} \\ \vdots & \vdots & \cdots & \vdots \\ \sum p_i(1-p_i)x_{Ki}x_{0i} & \sum p_i(1-p_i)x_{Ki}x_{1i} & \cdots & \sum p_i(1-p_i)x_{Ki}^2 \end{bmatrix}$$

```
for(r = 0; r < REGRESSORS; r++)
    for(i = 0; i < N; i++)
        for(c = 0; c < REGRESSORS; c++)
            result[r][c] += p[i]*(1-p[i])*x[i][r]*x[i][c];
```

Optimization

- Take advantage of locality in looping
- Openmp
  - Task parallelism: calculate first and second derivatives in parallel

$$\beta_{i+1} = \beta_i - \frac{\partial^2 l}{\partial \beta^2}^{-1} \frac{\partial l}{\partial \beta}$$

  - Openmp reduction clause cannot operate on array targets
- SSE3 Vector optimization

Linear Algebra

- Because inverting a matrix is costly, use Intel MKL's *LAPACKE_dgesv* to solve the linear system for the δ vector in the update step of each Newton-Raphson iteration

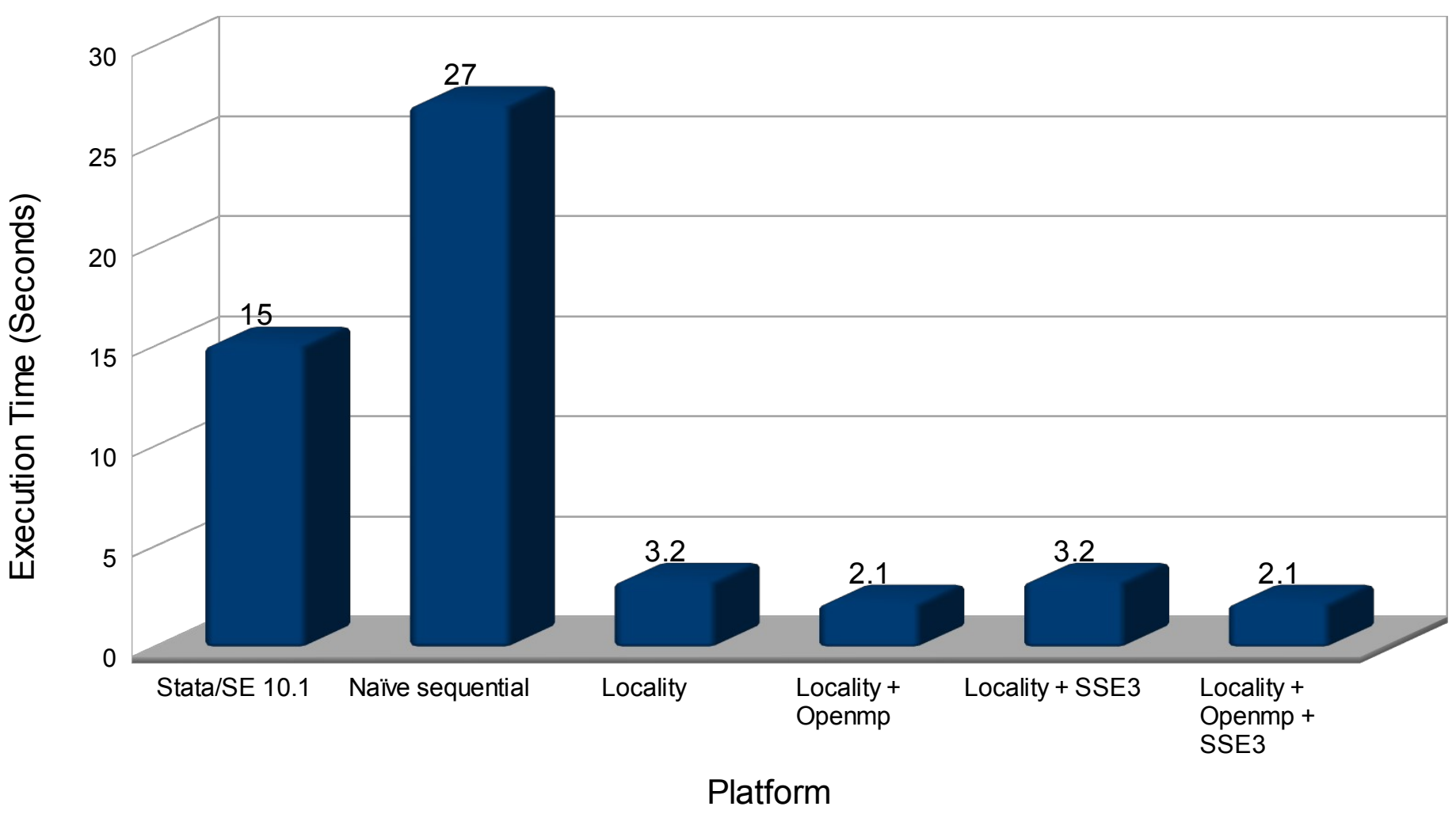
$$\frac{\partial^2 l}{\partial \beta^2}\begin{bmatrix}\delta_0\\ \delta_1\\ \vdots\\ \delta_K\end{bmatrix}=\frac{\partial l}{\partial \beta}$$

$$\beta_{i+1}=\beta_i-\delta$$

Platform

- Intel Core i3-530 2 cores, 4 threads
  - Private 32K + 32K L1 cache
  - Private 256K L2 cache
  - Shared 4MB L3 cache
- Ubuntu Linux 10.10 32-bit edition
  - Intel icc 11.1.073
  - Intel MKL 10.3.1.107 (linear algebra)
- Windows XP SP3
  - Stata/SE 10.1

Execution Time by Platform



Discussion

- My implementation was faster!
- Great benefit from locality optimization
- Small improvement from openmp
- No benefit from SSE3
- Suggestions welcome!