Jacobi Rotations for Singular Value Decomposition

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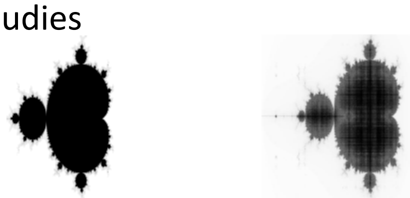
Introduction

- Singular Value Decomposition (SVD) is a technique for factoring matrices
- Examples where it is used:
  - Data Analysis
  - Visualization
  - Image Studies



Example 1: Image Compressed using SVD [1]

What is SVD?

- Factor matrix[2][3][4][5][6] A → $U*\Sigma*V^T$
  - U is size M*M and contains left singular values
  - Σ is size M*N and contains singular values of A
  - V is size N*N and contains the right singular values

$$\underset{A}{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}} = \underset{U}{\begin{bmatrix}-0.405 & -0.914\\-0.914 & 0.405\end{bmatrix}} * \underset{\Sigma}{\begin{bmatrix}5.465 & 0\\0 & 0.366\end{bmatrix}} * \underset{V^T}{\begin{bmatrix}-0.576 & -0.817\\0.817 & -0.576\end{bmatrix}}$$

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Mathematical Method

- Consider dense matrix A size MxN
- $U = A \times A^T$ & $V = A^T \times A$ ← EASY!!
- How to find singular values?
  - Find eigenvalues of U and V
  - Translation= Solve the following statement: $Ux = \lambda x$ & $Vy = \epsilon y$
  - Translation of Translation= Find c_x in these statements: $\det | U - c_1 I | = 0$ & $\det | V - c_2 I | = 0$
  - S = diagonal matrix with the square roots of c_x in the entries where $\sigma_1 > \sigma_2 > \ldots > \sigma_n$

  BAD!!! Don't even think about it

- Summary: Mathematical method is computationally bad, we need approximation methods:
  - Jacobi Rotations [3]
  - QR Decomposition [6]

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Jacobi Rotations

- $M^T M = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$
- $\Theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longrightarrow \Theta^T M^T M \Theta = D$
- Θ : off-diagonal components of M expressed as t = tan θ
- Setting t -> 0 it is can be expressed as t2 + 2ζt -1 =0
- ζ = (β-α)/2γ
- From the quadratic equation, cos θ and sin θ can be recovered[2][3]

SVD Algorithm

[3]Repeat

For all pairs i<j

$$\alpha = \sum_{k=1}^{n} U_{ki}^2 \qquad \zeta = (\beta - \alpha)/(2\gamma)$$

$$\beta = \sum_{k=1}^{n} U_{kj}^2 \qquad t = signum(\zeta)/(|\zeta| + \sqrt{(1+\zeta^2)})$$

$$c = 1/\sqrt{(1+t^2)}$$

$$\gamma = \sum_{k=1}^{n} U_{kj} * U_{ki} \qquad s = ct$$

← Compute Rotations

For k = 1 to n

$$t = U_{ki} \qquad t = V_{ki}$$

$$U_{ki} = ct - sU_{kj} \qquad V_{ki} = ct - sV_{kj}$$

$$U_{kj} = st + cU_{kj} \qquad V_{kj} = st + cV_{kj}$$

endfor

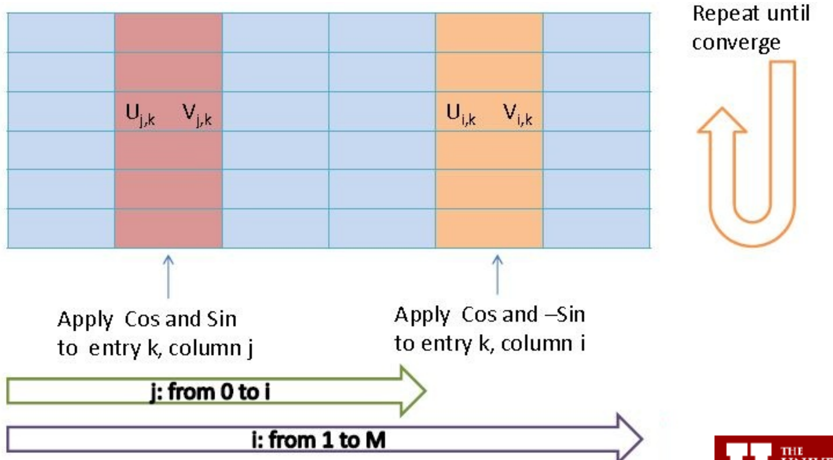
endfor

$until\ all\ |c|/\sqrt{(\alpha\beta)} \leq \epsilon$

Apply Rotations

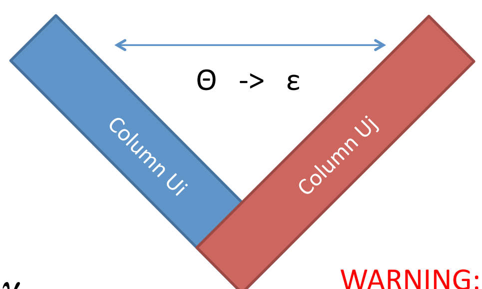
Graphical Detail

- U = A, V = I



Convergence

- Convergence tests

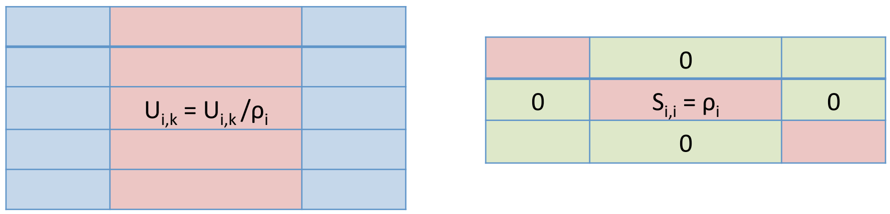


$$\theta = \frac{\gamma}{\sqrt{\alpha * \beta}}$$

WARNING:
Very sensitive point if you modify the algorithm and is mathematically wrong

S Matrix

- S = 0 Matrix (Green), New U (Blue)



$$\rho_i = \sqrt{\sum_{k=1}^{N} U_{i,k}}$$

References

1. CULA: "CUDA LApack and BLAS", www.culatools.com
2. M. T. Heath: "Scientific Computing: An Introductory Survey, 2nd Edition", Chapter 4
3. M. Sussman: "The Singular Value Decomposition", http://www.math.pitt.edu/~sussmanm/2071Spring08/lab09/index.html
4. G. W. Stewar: "On The Early History of the Singular Value Decomposition", http://conservancy.umn.edu/bitstream/1868/1/952.pdf
5. WolframAlpha: "Singular Value Decomposition", http://mathworld.wolfram.com/SingularValueDecomposition.html
6. HPECchallenge: "Singular Value Decomposition", http://www.ll.mit.edu/HPECchallenge/svd.html
7. NVIDIA: "GPU Computing SDK", http://developer.nvidia.com/gpu-computing-sdk