CS 4400
Computer Systems

LECTURE 4

Representing floats

Floating-point arithmetic

Floating Point

- Floating-point representation encodes rational numbers of the form $V = x \times 2^y$.
  - useful for numbers very large and very close to 0, why?
- Until the 1980s, there were many different conventions for how to represent floats and the operations on them.
  - accuracy not biggest concern, what was?
- Around 1985, IEEE Standard 754 surfaced as a carefully crafted standard for floating point.
  - by Kahan et al., now supported by virtually all computers

Fractional Numbers

- Decimal: $d_m\, d_{m\text{-}1} \cdots d_1\, d_0 . d_{\text{-}1}\, d_{\text{-}2} \cdots d_{\text{-}n}$ $\qquad d = \sum_{i=-n}^{m} 10^i \times d_i$
- Binary: $b_m\, b_{m\text{-}1} \cdots b_1\, b_0 . b_{\text{-}1}\, b_{\text{-}2} \cdots b_{\text{-}n}$ $\qquad b = \sum_{i=-n}^{m} 2^i \times b_i$
- *Example*: $101.11_2 = 2^2 + 2^0 + 2^{-1} + 2^{-2} = 5\ 3/4$
- What is the effect of shifting the binary point right/left?
- With finite-length encodings, there are decimal (and binary) fractions that cannot be represented exactly.
  - $1/3 = 0.33333\ldots_{10}$
  - $1/5 = 0.001100110011\ldots_2$

Clicker Question

If you have ResponseCard clicker, channel is **41**.

If you are using ResponseWare, session id is **CS1400U**.

Represent the value 51/32 as a binary number.

A. 0.010011

B. 0.100101

C. 1.100011

D. 1.100110

E. It cannot be represented exactly.

F. I don't know.

IEEE Floating-Point Representation

- Represents a number of the form $V = (-1)^s \times M \times 2^E$
- s: sign bit, interpretation for numeric value 0 is special
- E: exponent, weights by a power of 2
  - k bits (k=8 for single precision, k=11 for double), `exp` field
- M: significand, a fractional binary number
  - ranges [1, 2) or [0, 1), depending on whether the `exp` field is 0
  - n bits (n=23 for single precision, n=52 for double), `frac` field
- The value encoded by a given bit representation is divided into three cases, depending on the value of `exp`.

Case 1: Normalized Values

- Occurs when bit pattern of `exp` is neither all 0s nor all 1s.
- `exp` field interpreted as a signed integer in biased form
  - Bias = $2^{k\text{-}1} - 1$
  - Let e be the unsigned number represented by bits in `exp` field.
  - The actual exponent value is $E = e - (2^{k\text{-}1} - 1)$.
  - For double (k=11), $-1022 \leq E \leq 1023$. For single (k=8)?
- `frac` field interpreted as fractional value $0 \leq f < 1$
  - The significand value is $M = 1 + f$.
  - "Implied leading 1" representation gets additional bit for free
  - Thus, the range of M is [1,2).

Clicker Question

Recall: single precision uses 8 `exp` bits and 23 `frac` bits

$E = e - (2^{k-1} - 1)$, $M = 1 + f$, $V = (-1)^s \times M \times 2^E$

What is V for 0 01111111 00000000000000000000000?

A. 0

B. 0.5

C. 1

D. 2

E. It is not a normalized value.

Case 2: Denormalized Values

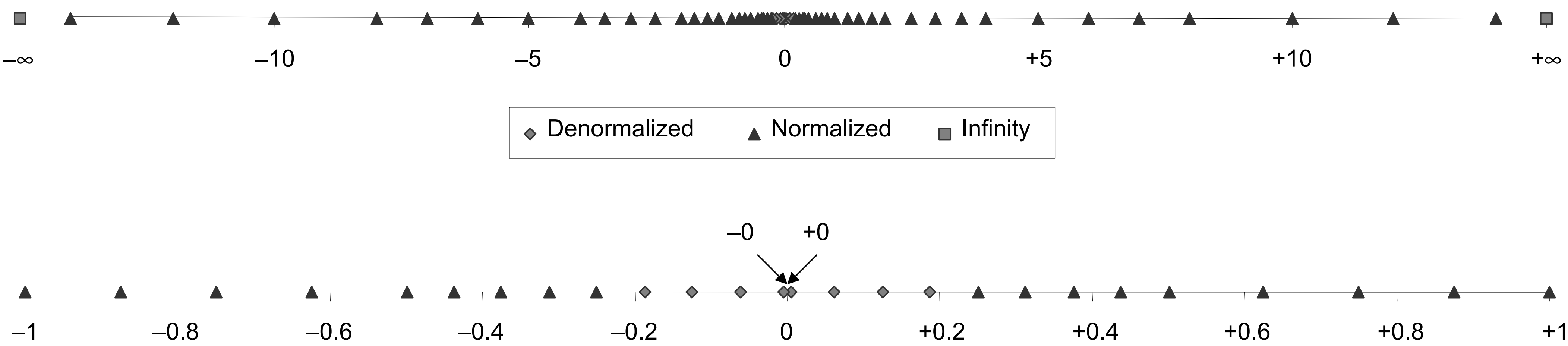
- Occurs when bit pattern of `exp` is all 0s (numeric value 0).
- The exponent value is $E = 1 - (2^{k-1} - 1)$.
- The significand value is $M = f$.
  - Without "implied leading 1".
  - Thus, the range of M is $[0,1)$.
- Why have denormalized numbers?
  - Can represent numeric value 0. Why cannot with normalized?
  - Can represent numbers very close to 0.
  - *Gradual underflow*—possible values are spaced evenly near 0.0.

Case 3: Special Values

- Occurs when bit pattern of `exp` is all 1s (numeric value 255 for single or 2047 for double).
- When the `frac` field is all 0s, the resulting value is ∞ (positive or negative, depending on *s*).
- When the `frac` field is nonzero, the resulting value is called a "NaN" (Not a Number).
  - Represents a result that cannot be given as a real number or as infinity.

Example: 6-bit Format

Assume a hypothetical 6-bit format with k=3 exponent bits and n=2 significand bits. What is the exponent bias?



- What are the normalized numbers with maximum magnitude?

  e = ? E = ? f = ? M = ? V = ?

- Are the representable numbers uniformly distributed?

Exercises: 5-bit Format

Assume a hypothetical 5-bit format with k=2 exponent bits and n=2 significand bits. The exponent bias is $2^{k-1} - 1 = 1$.

s e_1e_0 f_1f_0	e	E	f	M	V
0 00 00					
0 00 10					
0 01 01					
0 10 11					
0 11 00					
0 11 10					

Properties of IEEE Floating Point

- The value +0.0 always has a bit pattern of all 0s.
- The smallest denormalized value > 0 has a bit pattern consisting of 1 in LSB and all 0s elsewhere.
  - $M = f = 2^{-n}$, $E = 1 – (2^{k-1} – 1) = – 2^{k-1} + 2$
  - $V = M \times 2^{E} = 2$^($-n$-2^{k-1}+2)
- The largest denormalized value has a bit pattern consisting of an all-0 `exp` field and an all-1 `frac` field.
  - $M = f = 1$ – epsilon, $E = 1 – (2^{k-1} – 1) = – 2^{k-1} + 2$
  - $V = M \times 2^{E} =$ (1 – epsilon) × 2^(-2^{k-1}+2)

More Properties of IEEE FP

- The smallest normalized value > 0 has a bit pattern consisting of 1 in LSB of `exp` field and all 0s elsewhere.
  - $M = 1 + f = 1, E = e - (2^{k\text{-}1} - 1) = -2^{k\text{-}1} + 2$
  - $V = M \times 2^E = 2\text{^}(-2^{k\text{-}1}+2)$
- The value 1.0 has a bit pattern with all but the MSB of the `exp` field set to 1 and all other bits set to 0.
  - $M = 1 + f = 1, E = e - (2^{k\text{-}1} - 1) = 0$

More Properties of IEEE FP

- The largest normalized value has a bit pattern consisting of 0 in LSB of `exp` field and all 1s elsewhere.
  - $M = 1 - f = 2 -$ epsilon, $E = e - (2^{k-1} - 1) = 2^{k-1} - 1$
  - $V = M \times 2^E = (2 -$ epsilon$) \times 2$^$(2^{k-1} - 1)$

Rounding

- For a real value x, find the “closest” matching x' representable in floating-point format.
- The key problem is to define the direction to round a value that is halfway between two possibilities.
- Another approach is to determine representable values x^- and x^+ such that $x^- \leq x \leq x^+$ is guaranteed.
- IEEE floating-point format defines four rounding modes.
  - The default mode finds x'.
  - The other three can be used to compute x^- and x^+.

Rounding Modes

- *Round-to-even* (aka round-to-nearest) mode—default
  - rounds either upward or downward such that least-significant digit of the result is even, e.g., both \$1.50 and \$2.50 → \$2
- *Round-to-zero* mode
  - rounds positive numbers downward and negative numbers upward, giving value x'' such that $| x'' | \leq | x |$
- *Round-up* mode
  - rounds all numbers upward, giving value x^- such that $x^- \leq x$
- *Round-down* mode
  - rounds all numbers downward, giving value x^+ such that $x \leq x^+$

Floating-Point Operations

- The result of floating-point addition or multiplication is simply the exact result of the operation defined over real numbers, and then rounded (to be representable).
- Floating-point addition is not associative.
  - for single precision, `(3.14 + 1e10) – 1e10` is `0.0`
  - but, `3.14 + (1e10 - 1e10)` is `3.14`
- Floating-point multiplication is not associative or distributive over addition.
  - for single precision, `1e20 * (1e20 - 1e20)` is `0.0`
  - but, `1e20 * 1e20 – 1e20 * 1e20` is NaN

Clicker Question

True or False: In C, all `int` values can be represented as `float` values.

A. true

B. false

C. I don't know.

Floating Point in C

- Single precision: `float`, double precision: `double`
- Round-to-even mode
- C standard does not require IEEE format—no (standard) way to change rounding modes or get special values.
  - most systems provide access to such features, but details vary
- Casting among types changes numeric values as follows:
  - `int` to `float`: may be rounded
  - `int`/`float` to `double`: exact numeric value is preserved
  - `double` to `float`: may overflow or be rounded
  - `float`/`double` to `int`: truncated toward zero, may overflow

Clicker Questions

Always true? *Click* A: yes, B: no, C: I don't know.

Assume: `int x, float f, double d`

- `x == (int)(float)x`
- `x == (int)(double)x`
- `f == (float)(double)f`
- `d == (double)(float)d`
- `f == -(-f)`
- `2/3 == 2/3.0`
- `(d >= 0.0) || ((d*2) < 0.0)`
- `(d+f) - d == f`

Extended Precision

- Floating-point registers of the IA32 processors use 80-bit extended-precision format (with x87, not SSE).
  - k=15 exponent bits, n=63 fraction bits
- When normal single- and double-precision numbers are loaded from memory, they are converted to this format.
- Arithmetic is always performed in the extended format.
- Numbers are converted back to single- or double-precision as they are stored to memory
- Can lead to undesirable consequences (see text).

Summary: Representing Information

- Groups of bits are interpreted differently for integers, real numbers, and character strings.
  - encoding and byte-ordering conventions differ across machines
- C is designed to accommodate a wide range of word sizes and encodings.
  - most machines use two's complement and IEEE format
- In casting between signed and unsigned integers, the underlying bit patterns do not change.
- Due to finite encoding length, properties of computer arithmetic differ from those of integer/real arithmetic.

Summary: Representing Information

- *Overflow*—a result exceeds representable range.
- *Underflow*—a floating-point value is so close to 0.0, it is represented as such.
- Properties of computer arithmetic allow compilers to do many optimizations.
  - such as replacing `7*x` with `(x<<3)-x`
- Floating-point arithmetic must be used carefully because of its limited range and precision, as well as, because it does not obey some common math properties.