Error in “Linear Subspace Design for Real-Time Shape Deformation”

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16 February 2017 — A key contribution by Wang et al. [1] is noticing that *any* smoothness energy with affine functions in its null space will produce a deformation subspace fulfilling the “affine precision” property boasted by generalized barycentric coordinates.

In our *paper*, we construct a discrete smoothness energy by *squaring* a modified discrete Laplace operator. For a given mesh with n vertices, to measure the smoothness of a scalar function $\mathbf{x} \in \mathbb{R}^n$, we minimize:

$$E(\mathbf{x}) = \mathrm{tr}(\mathbf{x}^\mathsf{T} \underbrace{\mathbf{K}^\mathsf{T}\mathbf{M}^{-1}\mathbf{K}}_{\mathbf{Q}_{\text{paper}}} \mathbf{x}) \tag{1}$$

where $\mathbf{Q}_{\text{paper}} \in \mathbb{R}^{n\times n}$ is the discrete quadratic smoothness form (as described in the paper), $\mathbf{M} \in \mathbb{R}^{n\times n}$ is a typical mass matrix (e.g., lumped barycentric areas), and $\mathbf{K} \in \mathbb{R}^{n\times n}$ is constructed as the addition of the *usual* per-vertex discrete cotangent Laplacian $\mathbf{L} \in \mathbb{R}^{n\times n}$ and the sparse matrix computing normal derivatives at boundary vertices $\mathbf{N} \in \mathbb{R}^{n\times n}$:

$$\mathbf{K} := \mathbf{L} + \mathbf{N}. \tag{2}$$

This modified Laplacian $\mathbf{K}$ can be derived many different ways. This is not so surprising, as there are many ways of deriving the cotangent matrix $\mathbf{L}$. It is well known, that the vertex-based discrete Laplacian $\mathbf{L}$ can be constructed as a *projection* of the *edge-based* Crouzeix-Raviart discrete Laplacian $\mathbf{L}_{\text{cr}} \in \mathbb{R}^{k\times k}$ for a mesh with k edges (see, e.g., [2]):

$$\mathbf{L} = \mathbf{A}^\mathsf{T}\mathbf{L}_{\text{cr}}\mathbf{A}, \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{k\times n}$ is the incidence matrix that averages values on vertices to values on edges ($A_{ev} = ½$ if edge e is incident on vertex v, otherwise $A_{ev} = 0$). Similarly, the normal derivative matrix $\mathbf{N}$ is a projection of the Crouzeix-Raviart normal derivative matrix for edges $\mathbf{N}_{\text{cr}} \in \mathbb{R}^{k\times k}$:

$$\mathbf{N} = \mathbf{A}^\mathsf{T}\mathbf{N}_{\text{cr}}\mathbf{A}. \tag{4}$$

Using these, we can expand the energy described in the paper:

$$\mathbf{Q}_{\text{paper}} = \mathbf{A}^\mathsf{T}(\mathbf{L}_{\text{cr}} + \mathbf{N}_{\text{cr}})^\mathsf{T} \underbrace{\left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^\mathsf{T}\right)}_{\mathbf{B}_{\text{paper}}} (\mathbf{L}_{\text{cr}} + \mathbf{N}_{\text{cr}})\mathbf{A}, \tag{5}$$

where the parenthetical grouping suggests a possible *interpretation* of this energy as the integration of an edge-based *quantity* $((\mathbf{L}_{\text{cr}} + \mathbf{N}_{\text{cr}})\mathbf{A}\mathbf{x})$ via a non-standard “integration matrix” $\left(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^\mathsf{T}\right) =: \mathbf{B}_{\text{paper}} \in \mathbb{R}^{k\times k}$.

Via experimentation and confirmation with the original *code*, it is now clear that $\mathbf{B}_{\text{paper}}$ was replaced with the (simpler, diagonal) inverse Crouzeix-Raviart mass matrix $\mathbf{B}_{\text{code}} := \mathbf{M}_{\text{cr}}^{-1} \in \mathbb{R}^{k\times k}$.

The figures and results in [1] were created with code using a seemingly subtly *different* smoothness energy, constructed as:

$$\mathbf{Q}_{\text{code}} = \mathbf{A}^\mathsf{T}(\mathbf{L}_{\text{cr}} + \mathbf{N}_{\text{cr}})^\mathsf{T} \underbrace{\left(\mathbf{M}_{\text{cr}}^{-1}\right)}_{\mathbf{B}_{\text{code}}} (\mathbf{L}_{\text{cr}} + \mathbf{N}_{\text{cr}})\mathbf{A}. \tag{6}$$

This disparity is sometimes not noticeable qualitatively: on *some* meshes the resulting weights—and thus deformations—are very similar. However, on other meshes $\mathbf{Q}_{\text{paper}}$ has a strictly *larger* null space than just affine functions. This leads to sporadic behavior and failure to fulfill the “affine precision” property. Lacking a proof, $\mathbf{Q}_{\text{code}}$ on the other hand appears to be far more stable and *only* contains affine functions in its null space.

Since the behavior of $\mathbf{Q}_{\text{code}}$ is superior, the *paper* erroneously describes a different energy than used in the *code*.

1. Yu Wang, Alec Jacobson, Jernej Barbic, Ladislav Kavan. “Linear Subspace Design for Real-Time Shape Deformation”, *ACM SIGGRAPH*, 2015.
2. Miklós Bergou, Max Wardetzky, David Harmon, Denis Zorin, Eitan Grinspun. “A Quadratic Bending Model for Inextensible Surfaces”, *Symposium on Geometry Processing*, 2006.