OPTIMUM KINEMATIC DESIGN FOR A SEVEN DEGREE OF FREEDOM MANIPULATOR

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While today's general purpose manipulators contain six degrees of freedom, effective reduction in degrees of freedom occurs from singularity regions and from workspace obstacles. It is argued that the future general purpose manipulators will contain seven degrees of freedom, and that one particular kinematic design is superior to other possible 7 degree of freedom kinematic designs. Considerations of elimination of singularities, mechanical realizability, kinematic simplicity, and workspace shape are taken into account in arriving at this design.

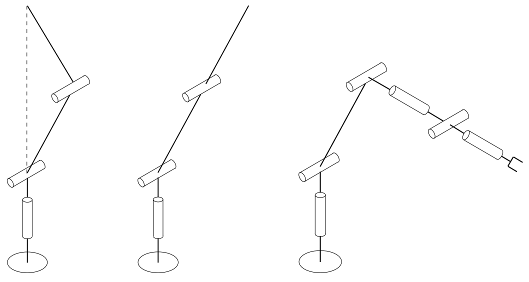
Current robot manipulators invariably are kinematically nonredundant, that is, they have at most the 6 degrees of freedom necessary for general positioning. The nonredundancy is associated with two difficulties that are not readily overcome: singularities and obstructed workspaces. Singularities are manipulator configurations where the degrees of freedom of the manipulator are reduced below 6, and arise from certain kinematic alignments of rotary joint axes. For example, the rotary manipulator without offsets in Fig. 1 (henceforth referred to as the 6R manipulator after the notation of [Pieper, 1968]) has three kinds of singularities [Featherstone, 1982], designated here as shoulder, elbow, and wrist singularities. For the shoulder singularity (Fig. 1A), the wrist is above the shoulder, making movements of the wrist perpendicular to the forearm–upper arm plane impossible. The elbow singularity (Fig. 1B) prevents radial movement of the wrist, which lies at the outer boundary of the workspace. The wrist singularity (Fig. 1C) aligns the roll axes of the forearm and hand, reducing by one the degrees of freedom.

Singularities are manifested in inverse kinematic problems where it is desired to find joint velocities that correspond to desired hand velocities. Joint velocities $\dot{\boldsymbol{\theta}}$ and hand velocities $\dot{\mathbf{x}}$ are related by the Jacobian matrix $\mathbf{J}$:

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\boldsymbol{\theta}}$$

where the 6-dimensional hand velocity $\dot{\mathbf{x}} = (\dot{\mathbf{p}}\ \boldsymbol{\omega})$ consists of a linear velocity vector $\dot{\mathbf{p}}$ of some hand reference point and an angular velocity vector $\boldsymbol{\omega}$, and where for an n degree of freedom manipulator $\dot{\boldsymbol{\theta}} = (\theta_1, \ldots, \theta_n)$. The Jacobian $\mathbf{J}$ is square for a 6 degree of freedom arm, but becomes noninvertible at a kinematic singularity. Desired Cartesian motions at the singular configuration become impossible to realize, manifested mathematically by the joint velocities evidently going to infinity. Near singularities the movements are also ill-conditioned, in the sense that while infinite joint velocities do not result still very large joint velocities are required. Thus there are forbidden regions represented by cones of ill-conditioned joint angles centered at the singular joint values [Paul and Stevenson, 1983]. As these cones are swept through configuration space by the more proximal joints, a substantial fraction of the workspace is lost. For this reason Paul and Stevenson characterize a kinematic 6 degree of freedom manipulator as functionally a 5 degree of freedom manipulator.

Singularity problems for 6 degree of freedom manipulators have long been recognized, and two main approaches have evolved to deal with them. One approach is to avoid singularities and associated ill-conditioned regions entirely by planning trajectories that stay away from them. Unfortunately it is not always possible to keep away from singularity points, and even if it were, a considerable burden is placed on the trajectory planner given the difficulty of relating the hand Cartesian space and joint space [Lozano-Perez, 1982]. The second approach is to violate the Cartesian motion specification, in effect achieving



(A) (B) (C)

Figure 1: Singularities arising from a 6 degree of freedom rotary manipulator. (A) The wrist lies along the first shoulder rotary axis in the shoulder singularity. (B) The elbow is straight in the elbow singularity. (C) The wrist is straight in the wrist singularity.

some compromise between desired hand motion and possible joint motion. Near singularities, for example, one can switch from Cartesian trajectories to joint-interpolated motion [Taylor, 1979]. Another possibility is to use generalized inverses to compromise selectively one of the Cartesian velocity components [Hanafusa, Yoshikawa, and Nakamura, 1981; Yoshikawa, 1983].

A second difficulty with nonredundant arms is workspace obstacles. Reaching around objects and through holes restricts the range of motion of joints and may effectively reduce the degrees of freedom.

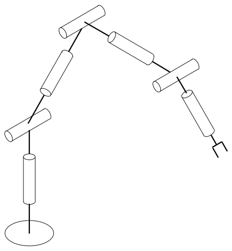


Figure 2: Manipulator 1: a 7 degree of freedom arm with a spherical shoulder joint, created by adding a roll motion in the upper arm link of the 6R manipulator.

Whole regions of the workspace may become unreachable if the manipulator degrees of freedom do not mesh with the task constraints. For example, the major motions of the upper arm and forearm for the 6R manipulator take place in a vertical plane. If an object obstructs primarily vertical plane movement, then it would not be possible for the manipulator to work behind the object.

The two difficulties of singularities and obstructed workspaces can be considerably ameliorated by adding degrees of freedom to a nonredundant manipulator. Internal singularities, which are inside the workspace boundary as for the shoulder and wrist singularities, could be virtually eliminated. Following Paul's dictum, a kinematic 7 degree of freedom manipulator would then function as a 6 degree of freedom manipulator, and hence would seldom manifest singularities.

Secondly, obstructions would be handled more easily. For example, the 7 degree of freedom manipulator with a spherical shoulder joint in Fig. 2, which is labeled manipulator 1 to distinguish it from alternative redundant designs considered later, has a self-motion describable by the rotation of the elbow point about a line joining the shoulder to wrist (Fig. 3). By a self-motion is meant an internal movement of the linkage that does not move the endpoint. For the 7 degree of freedom manipulator there is one degree of freedom in the self-motion, since six are constrained by the hand position; note that the self-motion requires movement in all the joints, though the relative joint motions are coupled. During the self-motion, the upper arm/forearm plane of motion is rotated about the shoulder/wrist line; thus the major plane of motion can be selected to avoid for example a predominantly vertical obstruction.

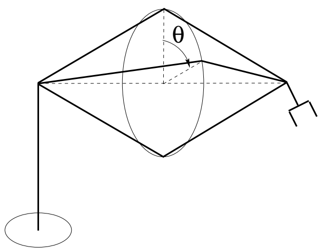


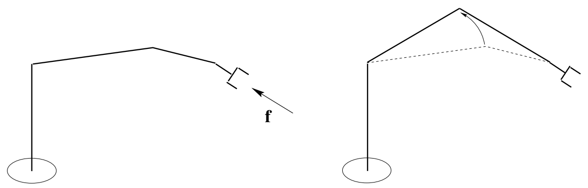
Figure 3: The self-motion of manipulator 1 corresponds to rotation of the elbow point about a line joining the shoulder to wrist. The angle θ between a normal line from the elbow to the shoulder-wrist line and this normal line when the elbow is straight up succinctly describes the redundancy.

Lastly, as an added bonus the kinematic redundancy would allow the minimization of torque loading at a weak joint. Suppose the manipulator has to absorb a force directed along the hand axis (Fig. 4A). Manipulator wrists are notoriously weak, and the external force could easily overcome the wrist motors. With a redundant arm, however, the forearm could be positioned through a self-motion so as to reduce the torque at the weakest wrist motor (Fig. 4B). The external force would be more completely absorbed by the more powerful proximal joints.

Kinematic Criteria for Redundant Arms

In designing geometries for a redundant manipulator, several criteria were set forth.

1. Elimination of internal singularities. It is taken here as an absolute prerequisite that the extra degrees of freedom eliminate the internal singularities at the shoulder and wrist. The external singularity at the elbow is of course inevitable because there is always an outer boundary of the workspace corresponding to maximum reach. If multiple degeneracies arise, it should be possible to exercise a self-motion of the redundant arm to find a configuration that avoids a singularity. In any case the redundancy should allow a planner to pass a trajectory through any point in the internal workspace without precondition.



(A) (B)

Figure 4: Torque loading at the wrist joint due to an external force along the hand can be minimized by selecting a self-motion that brings the forearm into an optimal alignment with the wrist.

2. Optimization of workspace. The extra joints should enhance as much as possible the ability to avoid obstacles and work in cluttered environments.

3. Kinematic simplicity. The ease of solving the kinematic equations aids path planning.

4. Mechanical constructability. The extra joint should lead to a facile mechanical design.

One of the most important geometries for 6 degree of freedom robot arms is the 6R manipulator of Fig. 1, which Roth [1980] argued as having a superior workspace relative to other geometries. The selection of a redundant arm is particularized here to the following design problem:

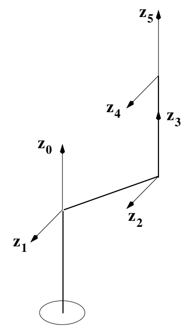


Figure 5: When θ_1 is set to 0, axes $\mathbf{x}_1$ and $\mathbf{x}_2$ are parallel and normal to the $\mathbf{z}_0/\mathbf{z}_1$ plane. Since $\mathbf{z}_2$ is also perpendicular to $\mathbf{x}_2$, it is also parallel to the $\mathbf{z}_0/\mathbf{z}_1$ plane.

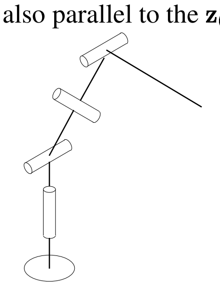


Figure 6: Creating a new link by adding a yaw motion within the upper arm link leads to a kinematically complex manipulator.

- *How is a rotary joint best added to the 6R manipulator to yield a 7 degree of freedom arm?*

The design goal is a 7 degree of freedom arm rather than an arm with 8 or more degrees of freedom, because as will be seen the 4 criteria above can be adequately matched by an arm with only one extra degree of freedom. As suggested above, multiple degeneracies lead to internal singularities in the optimum 7 degree of freedom arm that can be avoided, but adding more joints will never solve this problem (Appendix I). Hence the added complexity of even more degrees of freedom is not warranted.

The addition of a sliding joint rather than a rotary joint is ruled out because a sliding joint cannot eliminate the wrist singularity. Suppose the forearm is vertical, the wrist is straight, and θ_4 has been rotated so that $\mathbf{z}_4$ is parallel to $\mathbf{z}_1$ and $\mathbf{z}_2$ (Fig. 5). Then the $\mathbf{z}_3/\mathbf{z}_4$ plane is parallel to the $\mathbf{z}_0/\mathbf{z}_1$ plane, and since $\mathbf{z}_3$ and $\mathbf{z}_5$ are colinear all joint axes lie in the same plane. Thus this configuration represents a rotational singularity, because no component of hand angular velocity can lie normal to this plane. Adding a sliding joint can only change the hand linear velocity, not the hand angular velocity. Thus a sliding joint is incapable of eliminating this rotational singularity.

For generating possible arm geometries, the added joint is only allowed to increase the degrees of freedom at the shoulder, wrist, or elbow joint. Geometries that give rise to a totally new link of non-zero length and an associated spatially distinct joint, such as adding a yaw motion within the upper arm link (Fig. 6), are ruled out because of kinematic complexity [Pieper, 1968]. Three distinct possibilities that arise are analyzed below.

Manipulator 1: Spherical Shoulder Joint

By adding a roll motion in the upper arm link (Fig. 2), the 6R arm is given a spherical shoulder joint that makes it kinematically equivalent to the human arm (not counting shoulder movement, which adds several more degrees of freedom). An example of this manipulator appeared in [Takase, Inoue, and Sato, 1974]. The Denavit-Hartenberg parameters are listed in Table 1. Manipulator 1 has a regular kinematic structure of a roll axis on a link alternating with a pitch axis located at a joint; mechanically, this is an ideal configuration because consecutive motors are separated. The self-motions of this arm were analyzed in Fig. 3 and were shown to facilitate obstacle avoidance by selecting different planes of major motion. The criterion for kinematic simplicity is satisfied because the inverse kinematics is quite easily solved using wrist partitioning algorithms (Appendix II).

Joint i	a_i	s_i	α_i
1	0	s_1	$\pi/2$
2	0	0	$\pi/2$
3	0	s_3	$-\pi/2$
4	0	0	$\pi/2$
5	0	s_5	$-\pi/2$
6	0	0	$\pi/2$
7	0	0	0

Table 1: Denavit-Hartenberg parameters for manipulator 1.

Both the shoulder and wrist singularities as they arise from single degeneracies are eliminated. If the wrist is above the shoulder, it is now possible to move the wrist perpendicular to the major plane of motion. If the wrist is straight, the loss of rotary motion direction is compensated by the new joint. A double degeneracy can arise when the wrist is straight and the upper arm is vertical (Fig. 7), leading to a singularity. By rotating the elbow about the shoulder-wrist line, a configuration can be found where the upper arm roll axis and the forearm roll axis are made non-parallel with the base and hand roll axes respectively. This is the only situation where a trajectory planner must take special steps; the Cartesian point remains in the usable portion of the workspace.

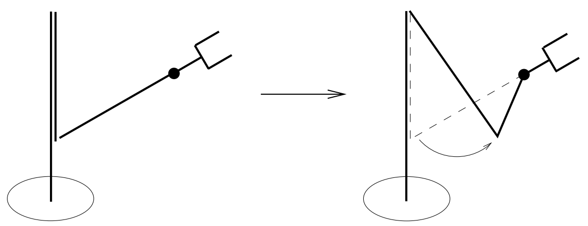


Figure 7: A double degeneracy for manipulator 1 arises when the wrist is straight and the upper arm is vertical. A self-motion brings the arm out of both degeneracies without moving the endpoint.

Manipulator 2: Two Degree of Freedom Elbow

In manipulator 1 a degree of freedom was added to the shoulder; for manipulator 2 a degree of freedom is added to the elbow. Essentially the only one way to add this degree of freedom without either duplicating the manipulator 1 kinematics or yielding a non-rational design is adding a yaw motion (Fig. 8); the Denavit-Hartenberg parameters are given in Table 2. First, it is a matter of viewpoint whether the shoulder in manipulator 1 is spherical or the elbow has two degrees of freedom, since the upper arm roll motion intersects the elbow as well; thus adding this roll motion does not give a distinct geometry. Second, adding a roll motion to the forearm duplicates the roll motion that is already there. This leaves a yaw motion acting after the pitch motion (the reverse is kinematically equivalent); a skew angle between these two rotary axes other than 90^o is not considered a rational design.

Manipulator 2 has the same self-motion as manipulator 1, so the workspace is equivalent, modulo the shape of the links which are swept differently. The inverse kinematics are as readily solved using a similar algorithm (Appendix III). Both the wrist and shoulder singularities are eliminated, except for the situation where the pitch elbow angle is 0^o and the yaw angle is 90^o (Fig. 9). If the wrist is then also straight, a double degeneracy results since 3 roll motions coincide. This configuration is avoidable,

Joint i	a_i	s_i	α_i
1	0	s_1	$\pi/2$
2	a_2	0	0
3	0	0	$-\pi/2$
4	0	0	$\pi/2$
5	0	s_5	$-\pi/2$
6	0	0	$\pi/2$
7	0	0	0

Table 2: Denavit-Hartenberg parameters for manipulator 2.

however, because a self-motion can be used to unstraighten the wrist and pitch the elbow.

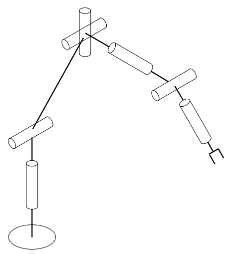


Figure 8: Manipulator 2: a two degree of freedom elbow joint created by adding a yaw motion to the elbow after the pitch motion.

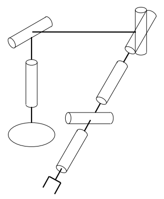


Figure 9: A double degeneracy arises for manipulator 2 when the elbow is at right angles due to a zero pitch angle and a 90 degree yaw angle coupled with a straight wrist. A self-motion can avoid this singularity.

Mechanically, this type of joint is not a desirable design. The motors interfere with each other, and a rather bulky joint would result even with offsets. Tendon actuation might improve the joint compactness, but 2 degree of freedom tendon actuated joints have proven difficult to design. The motor mass is also more distal than in manipulator 1. The advantage of alternate motor placements in links and joints is apparent when examining the manipulator 2 design.

Manipulator 3: Four Degree of Freedom Wrist

The last joint in the 6R arm whose degrees of freedom can be increased is the wrist, from 3 to 4. This is a common solution to the gimble lock problem for gyroscopes. Again, the only kinematically distinct way of increasing the wrist degrees of freedom is to add a yaw motion (Fig. 10); the Denavit-Hartenberg parameters are given in Table 3. While the wrist singularity is eliminated, the shoulder singularity remains. The self-motion moves no links, but merely changes the 4 wrist motor angles. Hence the workspace is not significantly improved; in general, a more proximal joint placement is required to improve workspace since greater volumes are swept out in configuration space. Mechanically this design suffers from the same problems as manipulator 2. The motor mass is moved even farther towards the distal end, significantly increasing the loading at the more proximal joints.

Joint i	a_i	s_i	α_i
1	0	s_1	$\pi/2$
2	a_2	0	0
3	0	0	$-\pi/2$
4	0	s_4	$\pi/2$
5	0	0	$-\pi/2$
6	0	0	$-\pi/2$
7	0	0	0

Table 3: Denavit-Hartenberg parameters for manipulator 3.

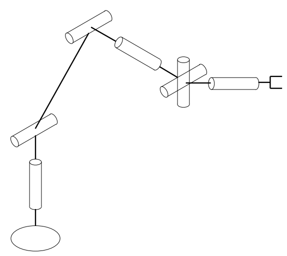


Figure 10: Manipulator 3: a yaw motion is added to the wrist after the pitch motion.

Conclusion

On balance manipulator 1 best satisfies the 4 criteria listed initially. Manipulator 2 suffers on the basis of mechanical design. Moreover, it could be argued that the double degeneracy of manipulator 2 is worse than the double degeneracy of manipulator 1, since it is more desirable to have bent elbows than upper arms straight up or down. Manipulator 3 must be rejected because it does not eliminate the shoulder singularity and it suffers from mechanical design difficulties.

Acknowledgments

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Appendix I: Persistence of Internal Singularities

It is proven in this appendix that for any rotary manipulator with arbitrary Denavity-Hartenberg [1955] parameters and nunbers of joints, there will always exist multiple degeneracies that lead to internal singularities[1]. Thus a trajectory planner will always face avoiding singular configurations and it is useless to add extra degrees of freedom in an attempt to completely eliminate internal singularities.

To proceed with the proof, the Denavit-Hartenberg link labeling is summarized first. Rotation axis $\mathbf{z}_{i-1}$ is located at joint i, which connects links $i-1$ and i; the angle at joint i is θ_i. The common normal between axes $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ defines $\mathbf{x}_i$, and the internal coordinate system for link i is located at the intersection of $\mathbf{z}_i$ and $\mathbf{x}_i$. Neighboring coordinate systems are related by three parameters:

a_i is the distance between $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ measured along $\mathbf{x}_i$,

s_i is the distance between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ measured along $\mathbf{z}_{i-1}$, and

α_i is the angle between $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ measured in a righthand sense about $\mathbf{x}_i$.

The rotation matrix $\mathbf{A}_i$ which transforms from link i coordinates to link $i-1$ coordinates is given by:

$$\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix}$$

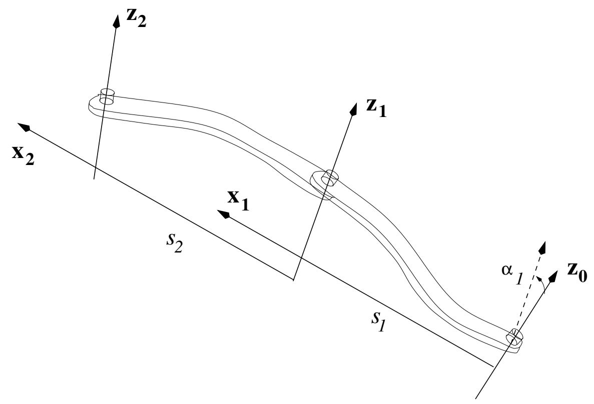

Figure 11: Joint angle θ_1 is set to zero, making $\mathbf{x}_2$ parallel to $\mathbf{x}_1$ and $\mathbf{z}_2$ parallel to the $\mathbf{z}_0/\mathbf{z}_1$ plane.

The basis for demonstrating the persistence of singularities is provided by the following lemma:

Lemma: For any arbitrary n-joint manipulator, it is possible to orient all joint axes parallel to the same plane.

Proof: Assume $\mathbf{z}_0$ and $\mathbf{z}_1$ are not colinear (otherwise find the first $\mathbf{z}_i$ not colinear with $\mathbf{z}_0$). Note that $\mathbf{x}_1$ is normal to the plane defined by $\mathbf{z}_0/\mathbf{z}_1$. All subsequent $\mathbf{z}_i$, $i > 1$, can be placed in the $\mathbf{z}_0/\mathbf{z}_1$ plane by the following induction argument.

- Set θ_2 to 0 or π. Then $\mathbf{x}_2$ is parallel to $\mathbf{x}_1$, hence is normal to the $\mathbf{z}_0/\mathbf{z}_1$ plane. Since $\mathbf{z}_2$ is perpendicular to $\mathbf{x}_2$, then $\mathbf{z}_2$ is parallel to the $\mathbf{z}_0/\mathbf{z}_1$ plane (Fig. 11).

- Suppose all joint angles from θ_2 to θ_{i-1} have been set to 0 or π. If θ_i is set to 0 or π, then $\mathbf{x}_i$ is made parallel to $\mathbf{x}_{i-1}$ and hence by induction is normal to the $\mathbf{z}_0/\mathbf{z}_1$ plane.

By induction, all joint axes can be made parallel to the same plane.

Definition: A rotational singularity is an arm configuration in which an arbitrary angular velocity of the hand cannot be specified.

From the Denavit-Hartenberg notation, the angular velocity $\boldsymbol{\omega}$ of the hand is:

$$\boldsymbol{\omega} = \sum_{i=1}^{n} \mathbf{z}_{i-1}\dot{\theta}_i$$

Theorem: For given orientations of the first two joint axes, there are 2^{n-2} rotational singularities.

Proof: By the Lemma, for each joint angle θ_i beyond the second, there are two angles 0 and π that align $\mathbf{z}_i$ with the $\mathbf{z}_0/\mathbf{z}_1$ plane. Thus for an n-joint manipulator, there are 2^{n-2} configurations in which all joint axes lie parallel to the same plane. Each such configuration leads to a rotational singularity since there can be no component of hand angular velocity normal to the $\mathbf{z}_0/\mathbf{z}_1$ plane.

Corollary: There are infinitely many internal rotational singularities.

Proof: Each value of θ_1 changes $\mathbf{z}_i$, hence the 2^{n-2} rotational singularities define a swept volume. Some of these singularities will be internal, corresponding to one or the other choice of 0 or π for each θ_i.

1 Appendix II: Inverse Kinematics for Manipulator 1

The solution of inverse kinematics for position is readily accomplished by modifying the wrist partitioning algorithm outlined in [Featherstone, 1983; Hollerbach and Sahar, 1983]. An alternative solution using a quadratic optimization rather than a specification of the self-motion is presented in [Benati, Morasso, and Tagliasco, 1982]. Suppose the redundancy is resolved by specification of the forearm-upper arm plane rotation θ about the shoulder/wrist line (Fig. 3).

Step 1: Find the wrist position. Given the position $\mathbf{p}$ of the hand reference point and the orientation matrix $\mathbf{R}$ of the hand, the wrist position $\mathbf{p}_w$ relative to the shoulder point is (Fig. 12):

$$\mathbf{p}_w = \mathbf{p} - \mathbf{R}\,{}^7\mathbf{p}_h - \mathbf{p}_s$$

where ${}^7\mathbf{p}_h$ is a vector from the wrist to the hand reference point specified in link 7 coordinates and $\mathbf{p}_s$ is a vector from the base to the shoulder.

Step 2a: Find the first 4 joint angles that would result if $\theta = 0$. This step is the same as in the 6R case, except with the new upper arm roll motion $\theta_3 = 0$. The results are modified directly from [Hollerbach and Sahar, 1983]:

$$\theta_1' = \tan^{-1}\left(\frac{p_{wy}}{p_{wx}}\right)$$

$$\sin\theta_4 = \frac{s_3^2 + s_5^2 - p_w^2}{2 s_3 s_5}$$

where the (') denotes a quantity referred to $\theta = 0$. Note that the elbow angle θ_4 is independent of θ. Having found θ_1', $\mathbf{p}_w$ can now be expressed in terms of link 1 coordinates, ${}^1\mathbf{p}_w'$. The first six rotational transformations have the form:

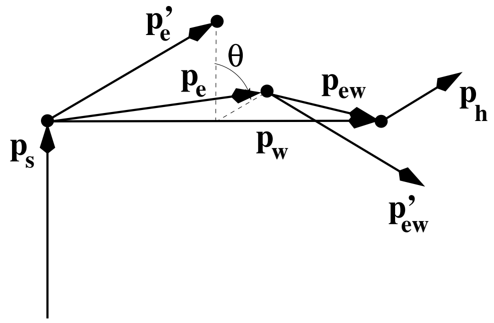

Figure 12: Inverse kinematic solution for Manipulator 1: Relative positions of the shoulder ($\mathbf{p}_s$), elbow ($\mathbf{p}_e$, with respect to the shoulder), wrist ($\mathbf{p}_w$, w.r.t. the shoulder, and $\mathbf{p}_{ew}$, w.r.t. the elbow), and hand ($\mathbf{p}_h$, w.r.t. the wrist). (') denotes a quantity refered to $\theta = 0$.

$$\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & 0 & \pm\sin\theta_i \\ \sin\theta_i & 0 & \mp\cos\theta_i \\ 0 & \pm 1 & 0 \end{bmatrix}$$

The shoulder pitch angle θ_2' is then found from ${}^1\mathbf{p}_w' = \mathbf{A}_1'^T \mathbf{p}_w$:

$$\sin\theta_2' = \frac{{}^1p_{wy}'(s_3 - s_5\sin\theta_4) - {}^1p_{wx}' s_5 \cos\theta_4}{s_3^2 + s_5^2 - 2 s_3 s_5 \sin\theta_4}$$

$$\cos\theta_2' = \frac{{}^1p_{wx}' + s_5\cos\theta_4\sin\theta_2'}{s_3 - s_5\sin\theta_4}$$

$$\theta_2' = \tan^{-1}\left(\frac{\sin\theta_2'}{\cos\theta_2'}\right)$$

Step 2b: Find the elbow position by rotating the elbow point about $\mathbf{p}_w$ by the angle θ. The elbow position corresponding to $\theta = 0$ is given by:

$$\mathbf{p}_e' = \mathbf{A}_1'\mathbf{A}_2' \begin{bmatrix} 0 \\ 0 \\ s_3 \end{bmatrix} = s_3 \begin{bmatrix} \cos\theta_1' \sin\theta_2' \\ \sin\theta_1' \sin\theta_2' \\ -\cos\theta_2' \end{bmatrix}$$

The Rodrigues formula can be used to find the new elbow position. Define $\mathbf{n} = \mathbf{p}_w/|\mathbf{p}_w|$. Then

$$\mathbf{p}_e = \mathbf{n}(\mathbf{n}\cdot\mathbf{p}_e')(1 - \cos\theta) + \mathbf{p}_e'\cos\theta + \mathbf{n}\times\mathbf{p}_e'\sin\theta$$

Step 2c: Find the first two joint angles from the elbow position. Note that the zero position of the upper arm is straight down.

$$\theta_1 = \tan^{-1}\left(\frac{p_{ey}}{p_{ex}}\right)$$

$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{p_{ex}^2 + p_{ey}^2}}{-p_{ez}}\right)$$

Step 2d: Find the roll angle θ_3. Consider the wrist position that would be located if the arm were placed by the (unprimed) angles θ_1, θ_2, and θ_4, with $\theta_3 = 0$. The vector from the elbow point $\mathbf{p}_e$ to this wrist point $\mathbf{p}'_{ew}$ is given by $\mathbf{p}'_{ew} = s_5 \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4 \, {}^4\mathbf{z}_4$. The actual forearm vector is given by $\mathbf{p}_{ew} = \mathbf{p}_w - \mathbf{p}_e$. Thus

$$\theta_3 = \tan^{-1}\frac{|\mathbf{p}'_{ew} \times \mathbf{p}_{ew}|}{\mathbf{p}'_{ew} \cdot \mathbf{p}_{ew}}$$

Step 3: Find the hand orientation relative to the forearm. The only difference from the 6R case is the added roll motion θ_3 which must be incorporated. Define $\mathbf{W}_j = \mathbf{A}_1 \ldots \mathbf{A}_j$ and ${}^i\mathbf{W}_j = \mathbf{A}_{i+1} \ldots \mathbf{A}_j$. Noting that $\mathbf{W}_7 = \mathbf{R}$ is the absolute orientation of the hand, then the relative hand orientation is given by:

$${}^4\mathbf{W}_7 = \mathbf{W}_4^T \mathbf{W}_7$$

Step 4: Find the wrist joint angles. This step is equivalent to that in [Hollerbach and Sahar, 1983] and is omitted.

Appendix III: Inverse Kinematics for Manipulator 2

The steps in the inverse kinematics for position is substantially the same as for manipulator 1. Where they differ, additional details are given.

Step 1: Find the wrist position.

Step 2a: Find the first 4 joint angles that would result if $\theta = 0$. Note that the pitch angle at the elbow is now θ_3, while the yaw angle $\theta_4 = 0$ when $\theta = 0$. Also θ_3 is now dependent on θ.

$$\sin\theta'_3 = \frac{a_2^2 + s_5^2 - p_w^2}{2a_2 s_5}$$

Step 2b: Find the elbow position by rotating the elbow point about $\mathbf{p}_w$ by the angle θ.

Step 2c: Find the first two joint angles from the elbow position. Since the zero position of the upper arm is horizontal,

$$\theta_2 = \tan^{-1}\left(\frac{p_{ez}}{\sqrt{p_{ex}^2 + p_{ey}^2}}\right)$$

Step 2d: Find the roll angle θ_3. Express $\mathbf{p}_{ew} = \mathbf{p}_w - \mathbf{p}_e$ in link 2 coordinates, given θ_1 and θ_2. Thus

$${}^2\mathbf{p}_e = \mathbf{A}_3\mathbf{A}_4(s_5 \, {}^4\mathbf{z}_4) = s_5 \begin{bmatrix} \cos\theta_3 \sin\theta_4 \\ \sin\theta_3 \sin\theta_4 \\ -\cos\theta_4 \end{bmatrix}$$

Solving,

$$\theta_3 = \tan^{-1}\left(\frac{{}^2p_{ewy}}{{}^2p_{ewx}}\right)$$

$$\theta_4 = \tan^{-1}\left(\frac{{}^2p_{ewx}\cos\theta_3 + {}^2p_{ewy}\sin\theta_3}{-\,{}^2p_{ewz}}\right)$$

Step 3: Find the hand orientation relative to the forearm.

Step 4: Find the wrist joint angles.

2 Footnote

1. This is an alternative proof to a theorem originally derived by Roger Brockett.

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