Input: X , $d : X \times X \to \mathbb{R}$

$s : X \times X \to [0, 1]$

L10: Spectral Clustering

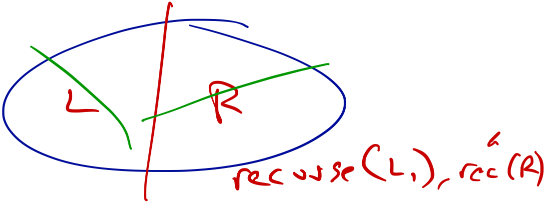
1. Hierarchical $\cancel{A}$C
   bottom-up

Jeff M. Phillips

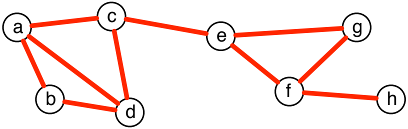
2. Assignment-based

February 13, 2019

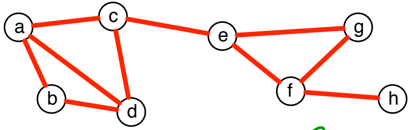
3. Hierarchy
   top-down



Graphs



Graphs

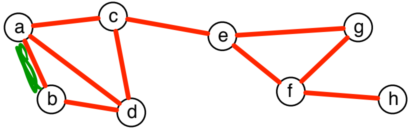


Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$

Graphs



$\alpha \in [0,1]$

e.g. $\alpha = 0.7$

$m \leq n^2$

$m \approx n^{1+\alpha}$

Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and $|V| = n$

$|E| = m$

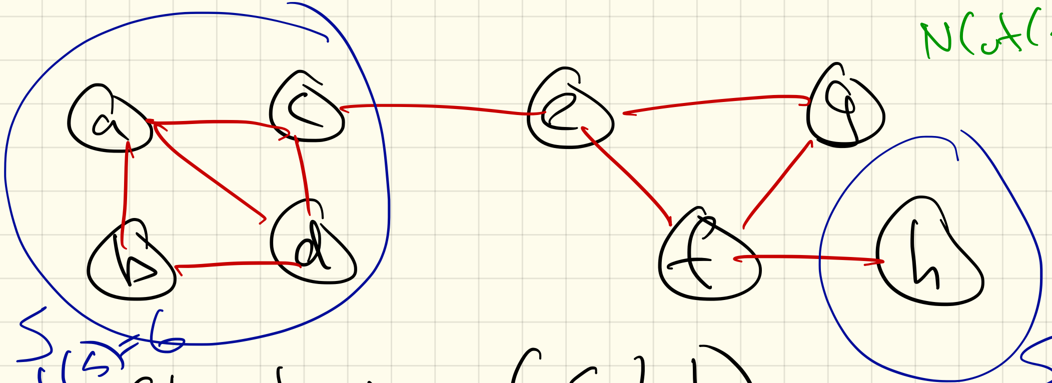
$$E = \Big\{\{a,b\},\{a,c\},\{a,d\},\{b,d\},\{c,d\},\{c,e\},\{e,f\},\{e,g\},\{f,g\},\{f,h\}\Big\}.$$

Matrix-Style: As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

$G =$

	a	b	c	d	e	f	g	h
a	0	1	1	1	0	0	0	0
b	1	0	0	1	0	0	0	0
c	1	0	0	1	1	0	0	0
d	1	1	1	0	0	0	0	0
e	0	0	1	0	0	1	1	0
f	0	0	0	0	1	0	1	1
g	0	0	0	0	1	1	0	0
h	0	0	0	0	0	1	0	0

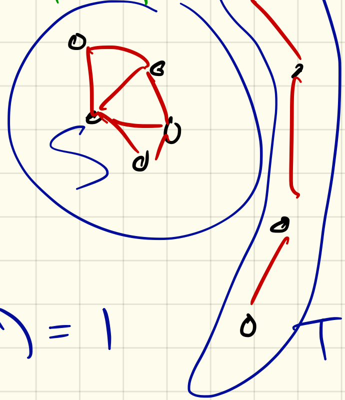
$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$vol(S) = 6$

$vol(S') = 1$

$N(cut(S')) = \frac{1}{1} + \frac{1}{9} = 1.1$



Clustering (Split)

$NCut(S) = \frac{1}{6} + \frac{1}{5} \approx 0.36$

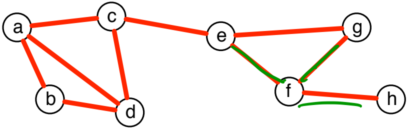
$V = S$ and $T = V \setminus S$

$Cut(S) = \#$ edges between $v \in S$ and $v' \in T$

$Vol(S) = \#$ edges $e \in E$ s.t. at least one endpoint of e is in S.

$$NCut(S) = \frac{Cut(S)}{Vol(S)} + \frac{Cut(S)}{Vol(V \setminus S)}$$

Laplacian Matrix



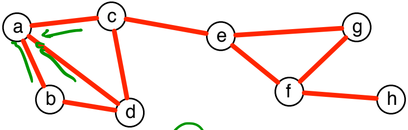
adjacency

degree

diagonal

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{array}{c c} & \begin{array}{cccccccc} & & & & & f & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ \\ f \\ \\ \\ \end{array} & \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Unnormalized Laplacian Matrix



$$L_0 = D - A = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

eigenvectors of L , v

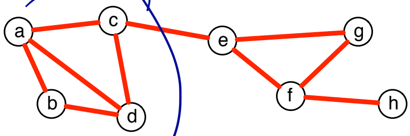
$$Lv = \lambda v$$

λ scalar

eigenvalue

if $\|v\| = 1$

Unnormalized Laplacian Matrix



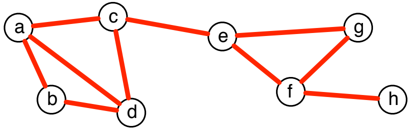
eigenvectors of L_0

Feidler

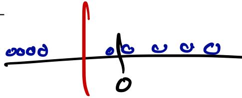
	v_1	v_2	v_3				v_7
λ	0	**0.278**	1.11	2.31	3.46	4	4.82
V_a	$1/\sqrt{8}$	$-.36$	0.08	0.10	0.28	0.25	$1/\sqrt{2}$
b	$1/\sqrt{8}$	$-.42$	0.18	0.64	$-.38$	0.25	0
c	$1/\sqrt{8}$	$-.20$	$-.11$	0.61	0.03	$-.25$	0
d	$1/\sqrt{8}$	$-.36$	0.08	0.10	0.28	0.25	$-1/\sqrt{2}$
	$1/\sqrt{8}$	0.17	$-.37$	0.21	$-.54$	$-.25$	0
	$1/\sqrt{8}$	0.36	$-.08$	$-.10$	$-.28$	0.75	0
	$1/\sqrt{8}$	0.31	$-.51$	$-.36$	$-.56$	0.56	0
	$1/\sqrt{8}$	0.50	0.73	0.08	0.11	0.11	0

eigs(L)

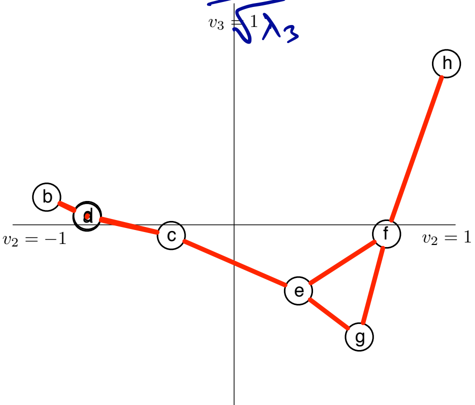
Unnormalized Laplacian Matrix



λ	**0.278**	1.11	
V	$-.36$	0.08	a
	$-.42$	0.18	b
	$-.20$	$-.11$	c
	$-.36$	0.08	d
	0.17	$-.37$	e
	0.36	$-.08$	f
	0.31	$-.51$	g
	0.50	0.73	h
	v_2	v_3	



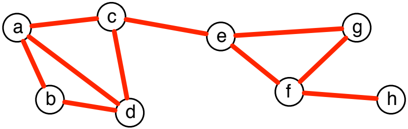
$$\frac{v_3(i)}{\sqrt{\lambda_3}}$$



x-axis $\dfrac{v_2(i)}{\sqrt{\lambda_2}}$

Laplacian Matrix

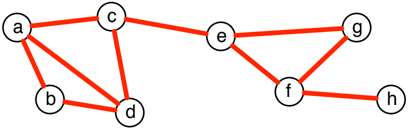
^Normalized



$$D^{-1/2} = \begin{pmatrix} 0.577 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.577 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.577 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.577 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} D_{11}^{-1/2} & & & \\ & D_{22}^{-1/2} & & \\ & & \ddots & \\ & & & D_{n,n}^{-1/2} \end{pmatrix}$$

Laplacian Matrix

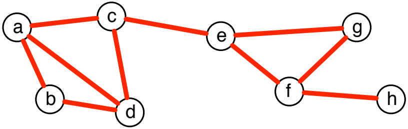


normalized Laplacian

$L = I - D^{-1/2}AD^{-1/2} =$ $D^{-1/2}(L_0)D^{-1/2} = D^{-1/2}(D-A)D^{-1/2}$

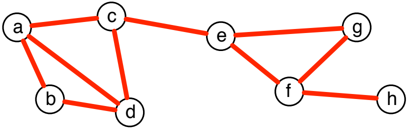
$$
\begin{pmatrix}
1 & -0.408 & -0.333 & -0.333 & 0 & 0 & 0 & 0 \\
-0.408 & 1 & 0 & -0.408 & 0 & 0 & 0 & 0 \\
-0.333 & 0 & 1 & -0.333 & -0.333 & 0 & 0 & 0 \\
-0.333 & -0.408 & -0.333 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.333 & 0 & 1 & -0.333 & -0.408 & 0 \\
0 & 0 & 0 & 0 & -0.333 & 1 & -0.408 & -0.577 \\
0 & 0 & 0 & 0 & -0.408 & -0.408 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.577 & 0 & 1
\end{pmatrix}.
$$

Laplacian Matrix



eigenvectors of L

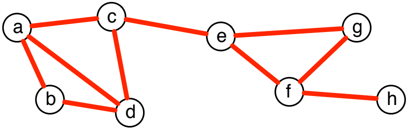
Laplacian Matrix



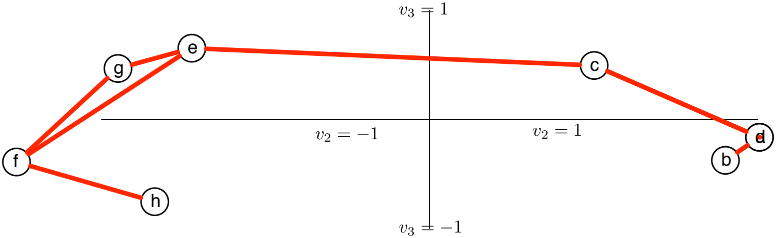
eigenvectors of L

λ	0	**0.125**	0.724	1.00	1.33	1.42	1.66	1.73
V	−.39	0.38	−.09	0.00	0.71	0.26	−.32	0.16
	−.32	0.36	−.27	0.50	0.00	−.51	0.38	−.18
	−.39	0.18	0.36	−.61	0.00	0.03	0.47	−.29
	−.39	0.38	−.09	0.00	−.71	0.26	−.32	0.16
	−.39	−.28	0.48	0.00	0.00	−.57	0.31	0.33
	−.39	−.48	−.29	0.00	0.00	0.05	−.31	−.65
	−.31	−.36	0.27	0.50	0.00	0.51	0.38	−.18
	−.22	−.32	−.61	−.35	0.00	−.07	0.27	0.51

Laplacian Matrix



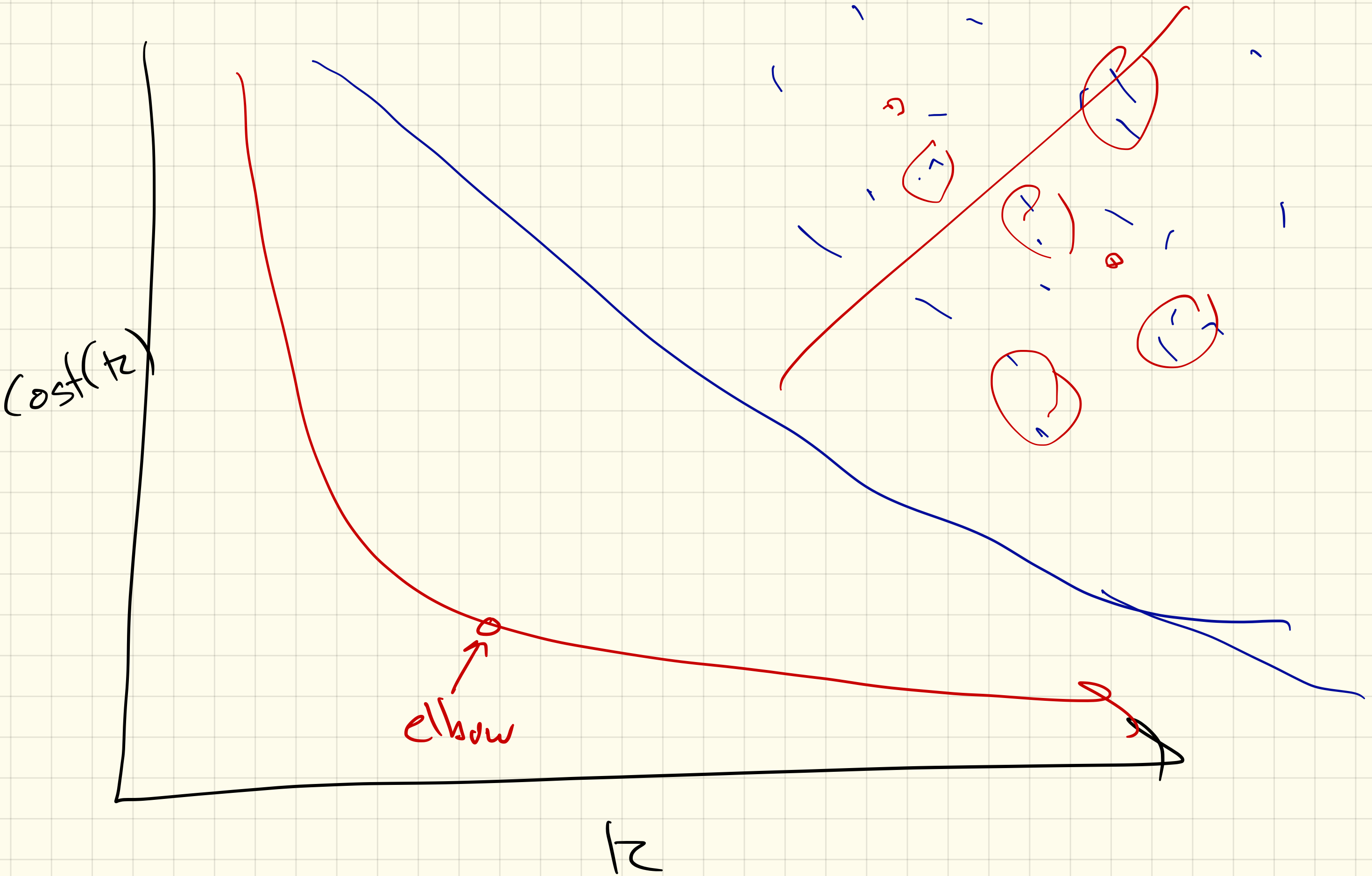
λ	**0.125**	0.724	
V	0.38	$-.09$	*a*
	0.36	$-.27$	*b*
	0.18	0.36	*c*
	0.38	$-.09$	*d*
	$-.28$	0.48	*e*
	$-.48$	$-.29$	*f*
	$-.36$	0.27	*g*
	$-.32$	$-.61$	*h*
	v_2	v_3	



Similarity s : X

A $n \times n$ matrix "affinity"

$$A_{ij} = s(x_i, x_j)$$



cost(k)
elbow
k