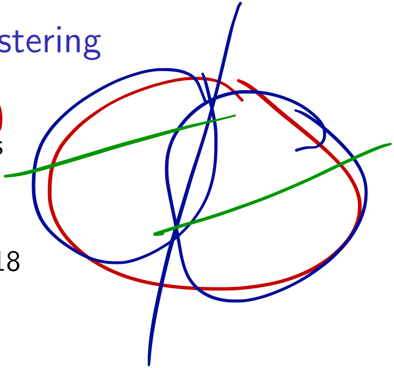
# L10: Spectral Clustering

(HAG)

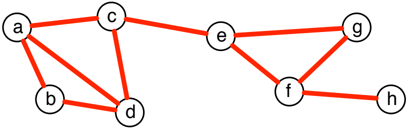
Jeff M. Phillips

February 12, 2018

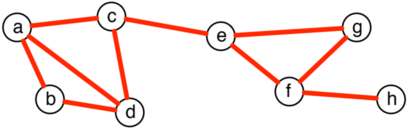
1. Bottom Up
2. Assignment
3. Top Down



# Graphs



# Graphs

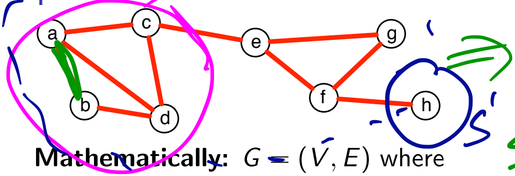



**Mathematically:** $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$$

# Graphs




$S, T \subset V$

$T = V \setminus S$

$S = \{a, b, c, d\}$

$T = \{e, f, g, h\}$

**Mathematically:** $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$

**Matrix-Style:** As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

$G =$

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $b$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $c$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| $d$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $e$ | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| $f$ | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 |
| $g$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $h$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$S \subset V$

$\text{Cut}(S) = \#\text{ edges } e = \{v_1, v_2\}$ s.t. $v_1 \in S \quad v_2 \notin S$

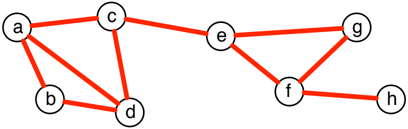
$\text{Vol}(S) = \#\text{ edges } e \in \{v_1, v_2\}$ s.t. $v_1$ or $v_2 \in S$

$$\text{NCut}(S) = \frac{\text{Cut}(S)}{\text{Vol}(S)} + \frac{\text{Cut}(V \setminus S)}{\text{Vol}(V \setminus S)}$$

$$\text{NCut}(S) = \frac{1}{6} + \frac{1}{5} = 0.367$$

$$\text{NCut}(S') = \frac{1}{1} + \frac{1}{10} = 1.1$$

# Laplacian Matrix



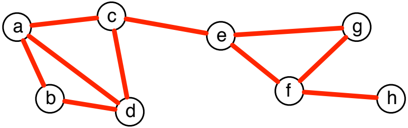

adjacency

degree

diagonal

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Unnormalized Laplacian Matrix




$$L_0 = D - A = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

eigenvectors, values of L

$v$ eigenvector of $M \in \mathbb{R}^{n \times n}$

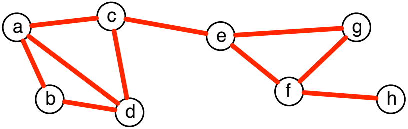
if $M v_i = \lambda_i v_i$

$\lambda$ = scalar ← eigenvalue

$n$ eigenvectors + values

# Unnormalized Laplacian Matrix

eigs( $L_0$ )




importance $\frac{1}{\sqrt{\lambda_i}}$

1d representation of G

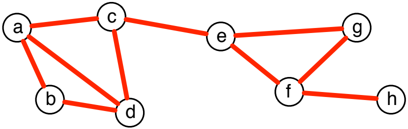
eigenvectors of $L_0$

| | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | | | | |
|---|---|---|---|---|---|---|---|
| $\lambda$ | 0 | **0.278** | 1.11 | 2.31 | 3.46 | 4 | 4.82 |
| $V$ | $1/\sqrt{8}$ | $-.36$ a | 0.08 | 0.10 | 0.28 | 0.25 | $1/\sqrt{2}$ |
| | $1/\sqrt{8}$ | $-.42$ | 0.18 | 0.64 | $-.38$ | 0.25 | 0 |
| | $1/\sqrt{8}$ | $-.20$ | $-.11$ | 0.61 | 0.03 | $-.25$ | 0 |
| | $1/\sqrt{8}$ | $-.36$ d | 0.08 | 0.10 | 0.28 | 0.25 | $-1/\sqrt{2}$ |
| | $1/\sqrt{8}$ | 0.17 | $-.37$ | 0.21 | $-.54$ | $-.25$ | 0 |
| | $1/\sqrt{8}$ | 0.36 | $-.08$ | $-.10$ | $-.28$ | 0.75 | 0 |
| | $1/\sqrt{8}$ | 0.31 | $-.51$ | $-.36$ | $-.56$ | 0.56 | 0 |
| | $1/\sqrt{8}$ | 0.50 | 0.73 | 0.08 | 0.11 | 0.11 | 0 |
| | $v_1$ | $v_2$ | $v_3$ | | | | |

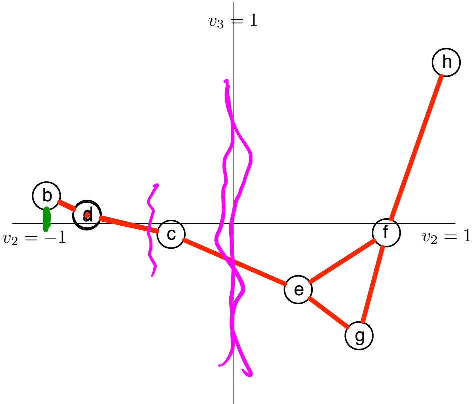
a
b
c
d
e
f
g
h

Feidler vector

# Unnormalized Laplacian Matrix

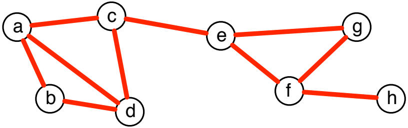



| $\lambda$ | **0.278** | 1.11 | |
|---|---|---|---|
| $V$ | $-.36$ | 0.08 | $a$ |
| | $-.42$ | 0.18 | $b$ |
| | $-.20$ | $-.11$ | $c$ |
| | $-.36$ | 0.08 | $d$ |
| | 0.17 | $-.37$ | $e$ |
| | 0.36 | $-.08$ | $f$ |
| | 0.31 | $-.51$ | $g$ |
| | 0.50 | 0.73 | $h$ |
| | $v_2$ | $v_3$ | |



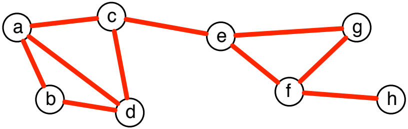

$b = (v_{1,b} \cdot \sqrt{\lambda_1}, 1$

# Laplacian Matrix




$$D^{-1/2} = \begin{pmatrix} 0.577 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.577 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.577 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.577 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Laplacian Matrix

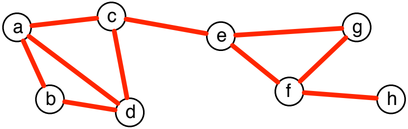



normalized Laplacian

$L = I - D^{-1/2} A D^{-1/2} =$

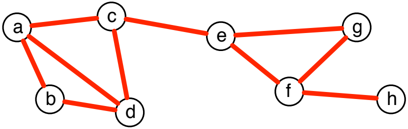
$$\begin{pmatrix} 1 & -0.408 & -0.333 & -0.333 & 0 & 0 & 0 & 0 \\ -0.408 & 1 & 0 & -0.408 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 1 & -0.333 & -0.333 & 0 & 0 & 0 \\ -0.333 & -0.408 & -0.333 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.333 & 0 & 1 & -0.333 & -0.408 & 0 \\ 0 & 0 & 0 & 0 & -0.333 & 1 & -0.408 & -0.577 \\ 0 & 0 & 0 & 0 & -0.408 & -0.408 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.577 & 0 & 1 \end{pmatrix}.$$

# Laplacian Matrix




eigenvectors of $L$

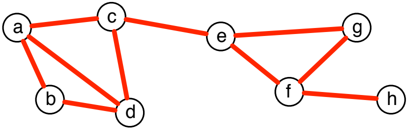
# Laplacian Matrix



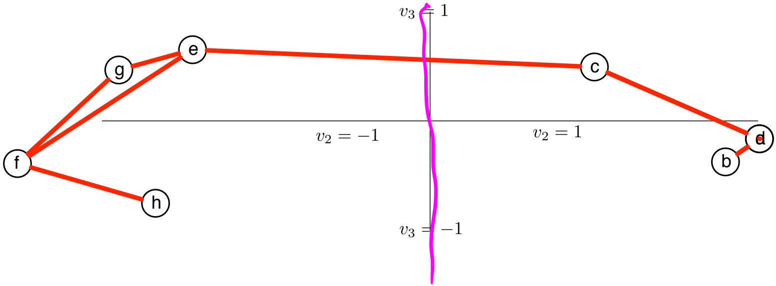

eigenvectors of $L$

| $\lambda$ | 0 | **0.125** | 0.724 | 1.00 | 1.33 | 1.42 | 1.66 | 1.73 |
|---|---|---|---|---|---|---|---|---|
| $V$ | −.39 | 0.38 | −.09 | 0.00 | 0.71 | 0.26 | −.32 | 0.16 |
| | −.32 | 0.36 | −.27 | 0.50 | 0.00 | −.51 | 0.38 | −.18 |
| | −.39 | 0.18 | 0.36 | −.61 | 0.00 | 0.03 | 0.47 | −.29 |
| | −.39 | 0.38 | −.09 | 0.00 | −.71 | 0.26 | −.32 | 0.16 |
| | −.39 | −.28 | 0.48 | 0.00 | 0.00 | −.57 | 0.31 | 0.33 |
| | −.39 | −.48 | −.29 | 0.00 | 0.00 | 0.05 | −.31 | −.65 |
| | −.31 | −.36 | 0.27 | 0.50 | 0.00 | 0.51 | 0.38 | −.18 |
| | −.22 | −.32 | −.61 | −.35 | 0.00 | −.07 | 0.27 | 0.51 |

# Laplacian Matrix




| $\lambda$ | **0.125** | 0.724 | |
|---|---|---|---|
| $V$ | 0.38 | $-.09$ | $a$ |
| | 0.36 | $-.27$ | $b$ |
| | 0.18 | 0.36 | $c$ |
| | 0.38 | $-.09$ | $d$ |
| | $-.28$ | 0.48 | $e$ |
| | $-.48$ | $-.29$ | $f$ |
| | $-.36$ | 0.27 | $g$ |
| | $-.32$ | $-.61$ | $h$ |
| | $v_2$ | $v_3$ | |



# Affinity Matrix

$$A = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & A_{ij} & & \end{bmatrix}$$

$$\mathrm{Sim}(v_i, v_j)$$