Compressed Sensing and OMP

Note Title 3/23/2016

Least Squares Regression

observations → $n \geq d$ ← variables

$$\underset{n\times d}{A}\ \underset{\substack{\text{unknown}\\ d\times 1}}{x} \approx \underset{n\times 1}{b} \implies \underset{x}{\arg\min} \|Ax - b\|_2^2$$

sparse

$$x = (A^TA)^{-1}A^Tb$$

if $d > n$ $\implies$ many solutions to x with 0 error

Compressed Sensing

$m \ll n$ non-zeros in X

[Tao - Candes] [Donaho] 2004

(model x) S ← signal, sparse $|S| = d$

non-zero s

$= [0, 1, 0, 0, 0, 1, 0, 0, \ldots, 0, 0]$

Measurement matrix $\underset{m\times d}{X}$ entry $X_{ij} \in \{-1, 0, +1\}$

integers

$\underset{m\times 1}{y} = XS^T$ ← actual measurements

measurement row X_i

$m \ll d \implies$ Recover S exactly

$m = K \cdot s \log\left(\frac{d}{s}\right)$

$K \in [4, 20]$

$X_i = [1, -1, 0, 0, -1, 0, -1, 1, 0 \ldots]$

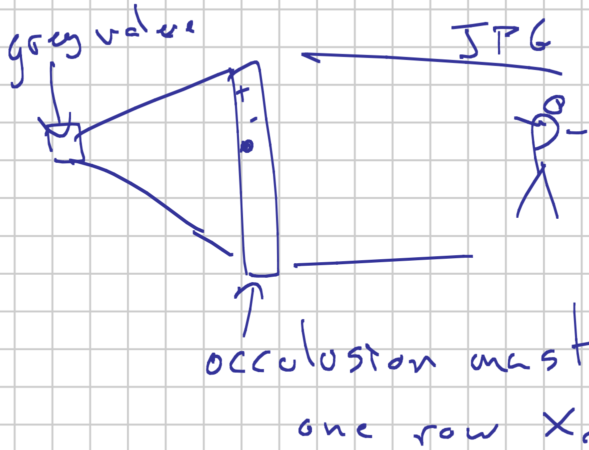
$y_i = x_i^T S = \langle x_i, S\rangle = (\cancel{1\cdot 0}) + (-1\cdot 1) + \cancel{0} + \cancel{0} + \cancel{0} + (-1, 1) + 0$

$= -2$

Examples CS

- single-pixel camera

  10 GPixels $\longrightarrow$ 2 GPixels



- Hubble Space Telescope
- MRI (on kids)

Given matrix X, mearsurments y

data A , b ← prediction

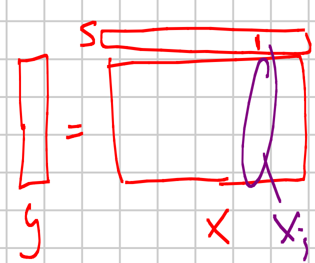
How do recover S?

x ← model

Orthogonal Matching Pursuit OMP

- choose column of X most useful

$$X_j = \arg\max_{X_j \in X} |\langle y, X_j \rangle|$$



- $$\gamma = \arg\min_{\gamma \in \mathbb{R}} \|y - X_j \gamma\| + \alpha\|\gamma\| \Rightarrow S_j = \gamma$$

$$\gamma = \frac{X_j^T y}{\langle X_j, X_j \rangle}$$

$y = X s^T$

- residual $r = y - X_j \gamma$

repeat with $y = r$

repeat until r all-zeros, or $\|r\|$ small enough t-th round

$$x_{j_1}, x_{j_2}, \ldots x_{j_t}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_t \end{pmatrix} = \underset{\gamma \in \mathbb{R}^t}{\arg\min} \left\| y - \begin{bmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_t} \end{bmatrix} \gamma \right\|_2 + \alpha \|\gamma\|_1$$

$$\gamma = (X_j^T X_j)^{-1} X^T y$$