Notes: Conditional Probability

CS 3130/ECE 3530: Probability and Statistics for Engineers

January 19, 2023

Review of “English translation” for events:

- \( A \cap B \) = “both events \( A \) and \( B \) happen”
- \( A \cup B \) = “either event \( A \) or \( B \) (or both) happens”
- \( A^c \) = “event \( A \) does not happen”

Set Theory Rules: (try drawing Venn diagrams of these)

- Definition of set difference: \( A - B = A \cap B^c \) “event \( A \) happens, but \( B \) does not”
- Associative Law:
  \[
  (A \cup B) \cup C = A \cup (B \cup C) \\
  (A \cap B) \cap C = A \cap (B \cap C)
  \]
- Commutative Law:
  \[
  A \cup B = B \cup A \\
  A \cap B = B \cap A
  \]
- Distributive Law:
  \[
  (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\
  (A \cap B) \cup C = (A \cup C) \cap (B \cup C)
  \]
- DeMorgan’s Law:
  \[
  (A \cup B)^c = A^c \cap B^c \\
  (A \cap B)^c = A^c \cup B^c
  \]

Probability Rules:

- Inclusion-Exclusion Rule: \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)
- Complement Rule: \( P(A^c) = 1 - P(A) \)
- Difference Rule: \( P(A - B) = P(A) - P(A \cap B) \)

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.

Conditional Probability:

\[
P(A | B) = \text{“the probability of event } A \text{ given that we know } B \text{ happened”}\]

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}
\]

Multiplication Rule:

\[
P(A \cap B) = P(A|B)P(B)
\]
Tree diagrams to compute “two stage” probabilities \((B = \text{first stage}, A = \text{second stage})\):

1. First branch computes probability of first stage: \(P(B)\)
2. Second branch computes probability of second stage, given the first: \(P(A \mid B)\)
3. Multiply probabilities along a path to get final probabilities \(P(A \cap B)\)

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?

<table>
<thead>
<tr>
<th>Picking a Box</th>
<th>Picking a Ball</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>1/2</td>
<td>1/3</td>
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<tr>
<td>1/2</td>
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<td>1/6</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Sampling without replacement:
I have a box with 10 red balls and 10 green balls. I draw 2 balls from the box without replacing them. What is the probability that I get 2 red balls?

Let \(R_1\) = “first ball red” and \(R_2\) = “second ball red” and use product rule:

\[
P(R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1) = \frac{1}{2} \times \frac{9}{19} = \frac{9}{38} \approx 0.24
\]

If I draw 3 balls without replacement, what is the probability that they are all red?

\[
P(R_1 \cap R_2 \cap R_3) = P(R_1 \cap R_2)P(R_3 \mid R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1)P(R_3 \mid R_1 \cap R_2)
\]

\[
= \frac{1}{2} \times \frac{9}{19} \times \frac{8}{18} = \frac{18}{171} \approx 0.11
\]