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Not a valid set definition: \[ C = \{1, 2, 3, 4, 2\} \]
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- Order in a set does not matter!

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- The “empty” or “null” set has no elements:
  \[ \emptyset = \{\} \]
Some Important Sets

- **Integers:**
  \[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]
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  \[ \mathbb{R} = \text{“any number that can be written in decimal form”} \]
  \[ 5 \in \mathbb{R}, \quad 17.42 \in \mathbb{R}, \quad \pi = 3.14159\ldots \in \mathbb{R} \]
Building Sets Using Conditionals

- Alternate way to define natural numbers:
  \[ N = \{ x \in \mathbb{Z} : x \geq 0 \} \]

- Set of even integers:
  \[ \{ x \in \mathbb{Z} : x \text{ is divisible by } 2 \} \]

- Rationals:
  \[ \mathbb{Q} = \{ p/q : p, q \in \mathbb{Z}, q \neq 0 \} \]
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Subsets

**Definition**
A set $A$ is a **subset** of another set $B$ if every element of $A$ is also an element of $B$, and we denote this as $A \subseteq B$.  

Examples:
- $\{1, 9\} \subseteq \{1, 3, 9, 11\}$
- $\mathbb{Q} \subseteq \mathbb{R}$
- $\{\text{apple, pear}\} \nsubseteq \{\text{apple, orange, banana}\}$
- $\emptyset \subseteq A$ for any set $A$
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A \textbf{sample space} is the set of all possible outcomes of an experiment. We’ll denote a sample space as $\Omega$. 

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**Examples:**

- **Coin flip:** $\Omega = \{H, T\}$
- **Roll a 6-sided die:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Pick a ball from a bucket of red/black balls:** $\Omega = \{R, B\}$
- **Tossing 2 coins?**
- **Shuffling deck of 52 cards?**
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An *event* is a subset of a sample space.
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- You roll a die and get an even number:
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- You flip a coin and it comes up "heads":
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- Your code takes longer than 5 seconds to run:
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Set Operations: Union

Definition

The **union** of two sets $A$ and $B$, denoted $A \cup B$ is the set of all elements in either $A$ or $B$ (or both).

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$A = \{1, 3, 5\}$ “an odd roll”

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Note: If $A \cap B = \emptyset$, we say $A$ and $B$ are **disjoint**.
Set Operations: Complement

| Definition | The complement of a set $A \subseteq \Omega$, denoted $A^c$, is the set of all elements in $\Omega$ that are not in $A$. |

Example:

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Set Operations: Difference

Definition

The **difference** of a set $A \subseteq \Omega$ and a set $B \subseteq \Omega$, denoted $A - B$, is the set of all elements in $\Omega$ that are in $A$ and are not in $B$.

Example:

$A = \{3, 4, 5, 6\}$

$B = \{3, 5\}$

$A - B = \{4, 6\}$

Note: $A - B = A \cap B^c$
DeMorgan’s Law

Complement of union or intersection:

\[
(A \cup B)^c = A^c \cap B^c
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What is the English translation for both sides of the equations above?
Exercises

Check whether the following statements are true or false. 
(Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
Probability

Definition

A **probability function** on a finite sample space \( \Omega \) assigns every event \( A \subseteq \Omega \) a number in \([0, 1]\), such that

1. \( P(\Omega) = 1 \)
2. \( P(A \cup B) = P(A) + P(B) \) when \( A \cap B = \emptyset \)

\( P(A) \) is the **probability** that event \( A \) occurs.
Equally Likely Outcomes

The number of elements in a set $A$ is denoted $|A|$. 

Example: Rolling a 6-sided die

$P(\{1\}) = \frac{1}{6}$

$P(\{1, 2, 3\}) = \frac{1}{2}$
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If $\Omega$ has a finite number of elements, and each is equally likely, then the probability function is given by

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Repeated Experiments

If we do two runs of an experiment with sample space $\Omega$, then we get a new experiment with sample space

$$\Omega \times \Omega = \{(x, y) : x \in \Omega, y \in \Omega\}$$
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The element \((x, y) \in \Omega \times \Omega\) is called an **ordered pair**.
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Properties:
Order matters: $(1, 2) \neq (2, 1)$
Repeats are possible: $(1, 1) \in \mathbb{N} \times \mathbb{N}$
More Repeats

Repeating an experiment $n$ times gives the sample space

$$\Omega^n = \Omega \times \cdots \times \Omega \quad (n \text{ times})$$

$$= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}$$
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The element \((x_1, x_2, \ldots, x_n)\) is called an \textbf{n-tuple}.

If \(|\Omega| = k\), then \(|\Omega^n| = k^n\).
Probability Rules

Complement of an event $A$

\[ P(A^c) = 1 - P(A) \]

Union of two overlapping events $A \cap B \neq \emptyset$

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
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$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number’s digits is 5
Permutations

A permutation is an ordering of an $n$-tuple. For instance, the $n$-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$

$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

The number of unique orderings of an $n$-tuple is $n$ factorial:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2$$
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How many ways can you rearrange \((1, 2, 3, 4)\)?