Prob Stats

Joint Continuous RVs

Mar 14, 2023

$$\binom{n}{k} p^{k} (1-p)^{n-k}$$

$n=100 \quad k=20$

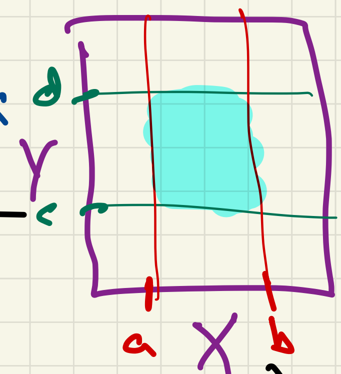
$$\binom{100}{20} 0.18^{20} (0.82)^{80} = \Pr(\ldots)$$

Joint Distributions for Continuous RV.

for 1 RV X

pdf $f_X(x) \geq 0$ $\quad \int_{x=-\infty}^{\infty} f_X(x)\,dx = 1$

$$P_r(\{a \leq X \leq b\}) = \int_{x=a}^{b} f(x)\,dx$$



joint pdf $\Longrightarrow$ 2 R.V. X, Y

$f_{X,Y}(x,y)$

$$P_r(\{a \leq X \leq b\} \cap \{c \leq Y \leq d\})$$
$$= \int_{y=c}^{d} \int_{x=a}^{d} f_{X,Y}(x,y)\,dx\,dy$$

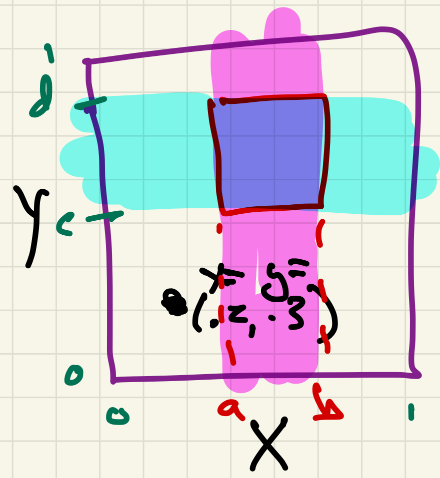
2 contin R.V. X, Y

joint pdf $f_{X,Y}(x,y) = f(x,y)$

① $f(x,y) \geq 0 \quad \forall x,y$

② $$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

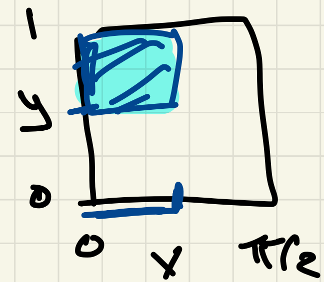
③ $$\Pr\left(\{a < X < b\} \cap \{c < Y < d\}\right) = \int_{y=c}^{d} \boxed{\int_{x=a}^{b} f(x,y)\,dx}\,dy$$



2 RVs X, Y

pdf: $f(x,y) = \begin{cases} 2y\sin(x) & 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1 \\ 0 & \text{o.w} \end{cases}$

$$\Pr\left(0 \le X \le \frac{\pi}{4},\ 0.5 \le Y \le 1\right)$$



$$= \int_{y=0.5}^{1} \int_{x=0}^{\pi/4} 2y\sin(x)\,dx\,dy$$

$$= \int_{y=0.5}^{1} -2y\cos(x) \Big|_{x=0}^{\pi/4} dy$$

$$= \int_{y=0.5}^{1} \left(-2y\left(\sqrt{2}/2 - 1\right)\right) dy$$

$$= -y^2\left(\sqrt{2}/2 - 1\right) \Big|_{y=0.5}^{1}$$

$$= -\left(1 - \frac{1}{4}\right)\left(\sqrt{2}/2 - 1\right) = \frac{6 - 3\sqrt{2}}{8} \simeq 0.2197$$

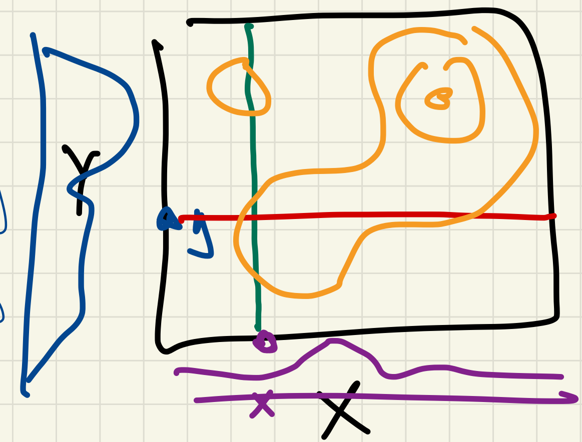
Marginal Probabilities

$f_{X,Y} = f$

$f_{X,Y} \Rightarrow f_X$ or f_Y

$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$

$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$



$f_Y(y) = 2y$

$f_X(x) = \sin(x)$

example $f(x,y) = 2y\sin(x)$ $\quad x \in (0, \pi/2)$, $y \in [0,1]$

$\boxed{f_X(x)} = \int_{y=-\infty}^{\infty} 2y\sin(x)\,dy = y^2\sin(x)\Big|_{y=0}^{1} = (1-0)\sin(x) = \boxed{\sin(x)}$

$f_Y(y) = \int_{x=-\infty}^{\infty} 2y\sin(x)\,dx = -2y\cos(x)\Big|_0^{\pi/2} = -2y(0-1) = 2y$

Conditional Probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

conditional probability of X given $Y = y$

pdf $\boxed{f(x \mid Y=y) = \frac{f(x,y)}{f_Y(y)}}$ or $f(x \mid Y=y) = 0$ if $f_Y(y) = 0$

$$\Pr(a \le X \le b \mid Y=y) = \int_{x=a}^{b} f(x \mid Y=y)\, dx$$

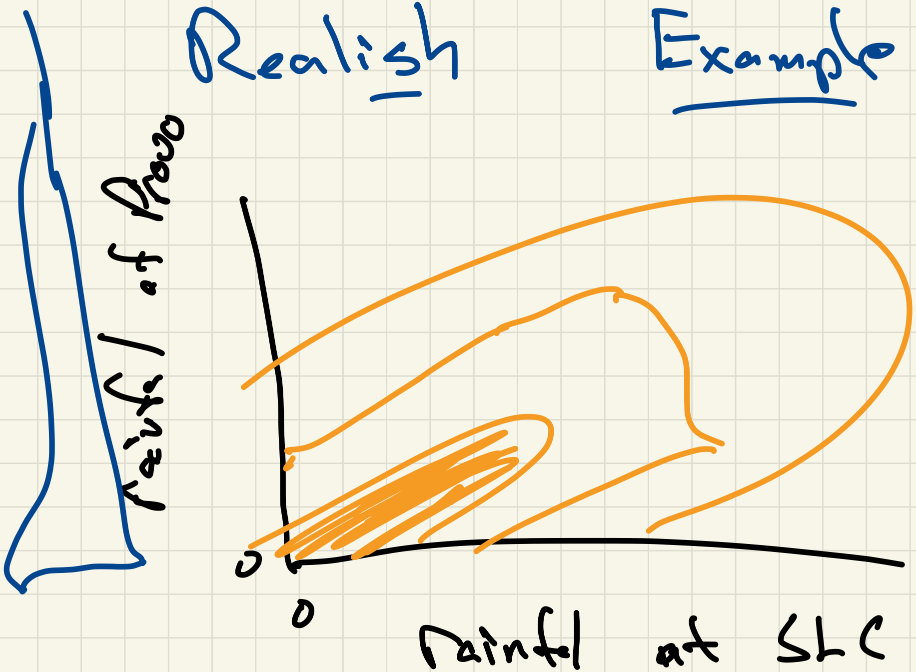
$$\boxed{f(x,y) = 2y\sin(x)}$$

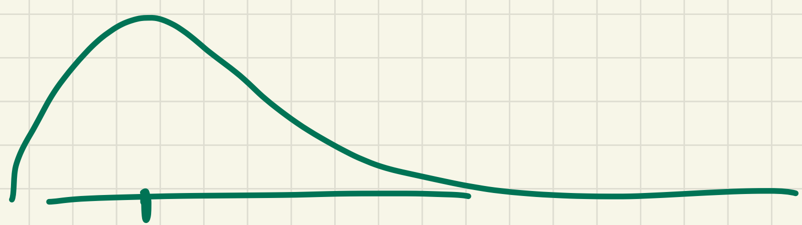
$$f(x \mid Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{2y\sin(x)}{2y} = \boxed{\sin(x)}$$

Note

this may depend on y.

Realish Example





Independence

joint contin R.V. X, Y are **independent** if

① $f(x,y) = f_X(x) \cdot f_Y(y)$ $\quad \forall x, y$

② $f(x \mid Y=y) = f_X(x)$ $\quad \forall x, y$

$\sin(x) = \sin(x)$

③ $f(y \mid X=x) = f_Y(y)$ $\quad \forall x, y$

equivalent

Conditional Expectation

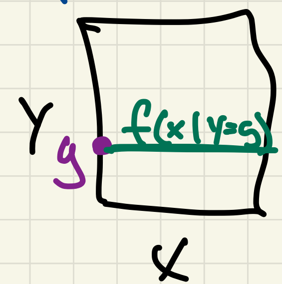
R.V. X, Y

$\boxed{\text{condition } Y = y}$

$f_{X,Y}(x,y) = f(x,y)$

$$E[X] = \int_{x=-\infty}^{\infty} x \, f_X(x)\, dx$$

$$E[X \mid Y = y] = \int_{x=-\infty}^{\infty} x \, \underline{f(x \mid Y=y)} \, dx$$



$$f(x,y) = x^2 + \frac{4}{3}xy + y^2$$

$(x,y) \in [0,1]^2$

$Y = \frac{1}{2}$

$$f(x \mid y=\tfrac{1}{2}) = \frac{x^2 + \frac{2}{3}x + \frac{1}{4}}{(11/12)}$$

$$E[X \mid Y=\tfrac{1}{2}] = \int_{x=0}^{1} x \, \frac{x^2 + \frac{2}{3}x + \frac{1}{4}}{11/12} \, dx$$

$$= \frac{12}{11} \cdot \left(\frac{x^4}{4} + \frac{2x^3}{9} + \frac{x^2}{8} \right) \Big|_0^1$$

$$= \frac{43}{66}$$