L9: Linear Algebra Review pt 2

the quickening?

Norms, Linear Independence

+ Rank

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FoDA

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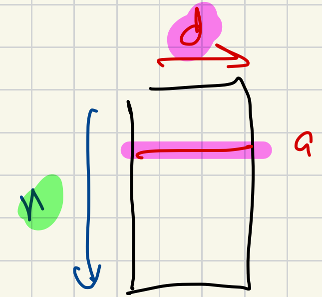
Linear Algebra

Vectors & Matrices

$u, v, x, a_i \in \mathbb{R}^d$

$u = (u_1, u_2, \dots, u_d)$

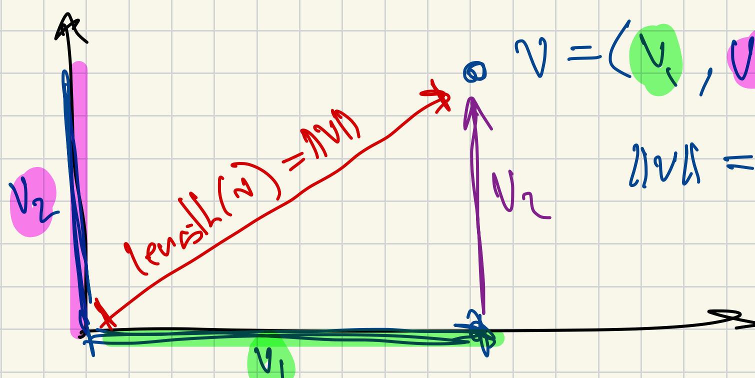
$A, B \in \mathbb{R}^{n \times d}$



Norm

$\text{length}(v) = \|v\|$ (latex \|)

$d = 2$



$v = (v_1, v_2) \in \mathbb{R}^2$

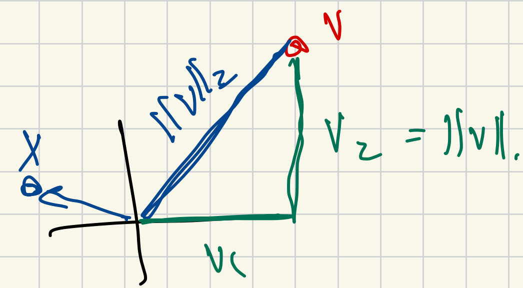
$\|v\| = \sqrt{v_1^2 + v_2^2}$

Pythagorean

Vector $v \in \mathbb{R}^d$ $\quad v = (v_1, v_2, \dots, v_d)$

Norm $\|v\| = \sqrt{v_1^2 + v_2^2 + \dots + v_d^2} = \|v\|_2$

Euclidean $= \sqrt{\sum_{j=1}^{d} v_j^2}$



p-Norm $\|v\|_p = \left(\sum_{j=1}^{d} |v_j|^p \right)^{1/p}$

$p \in [1, \infty) + \infty$

$\|v\|_1 = \left(\sum_{j=1}^{d} |v_j|^1 \right)^{1/1} = \sum_{j=1}^{d} |v_j|$

$x = (-3, 2, -1)$

$\|x\|_1 = 3 + 2 + 1 = 6$

$\|x\|_2 = \sqrt{9 + 4 + 1} = \sqrt{14}$

$$\boxed{\|v\|_2^2 = \sum_{j=1}^{d} (v_j)^2 = \langle v, v \rangle = \sum_{j=1}^{d} v_j \cdot v_j}$$

Matrix Norm

$$A \in \mathbb{R}^{n \times d}$$

In world of 2-Norms

$$A = \begin{bmatrix} \text{---} \; a_1 \; \text{---} \\ \text{---} \; a_2 \; \text{---} \\ \text{---} \; a_3 \; \text{---} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & & \\ & & A_{nd} \end{bmatrix}$$

Frobenius

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \boxed{\sum_{j=1}^{d} (A_{ij})^2}}$$

unrolling the matrix into vector

$$= \sqrt{\sum_{i=1}^{n} \boxed{\|a_i\|_2^2}}$$

Spectral

eigenvectors
values

$$\|A\|_2 = \max_{\substack{x \in \mathbb{R}^d \\ x \neq 0}} \frac{\|\underbrace{Ax}_{= y \in \mathbb{R}^n}\|_2}{\|x\|_2}$$

$$\|A\|_F \geq \|A\|_2$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$\|A\|_F^2 = 9 + 49 + 4 + 1 + 4 + 25$$

$$= 92$$

$$\|A\|_F = \sqrt{92} \approx 9.5$$

Linear Independence

fix $\mathbb{R}^d$

set $X =$

k vectors $\{x_1, x_2, \ldots x_k\} \in \mathbb{R}^d$

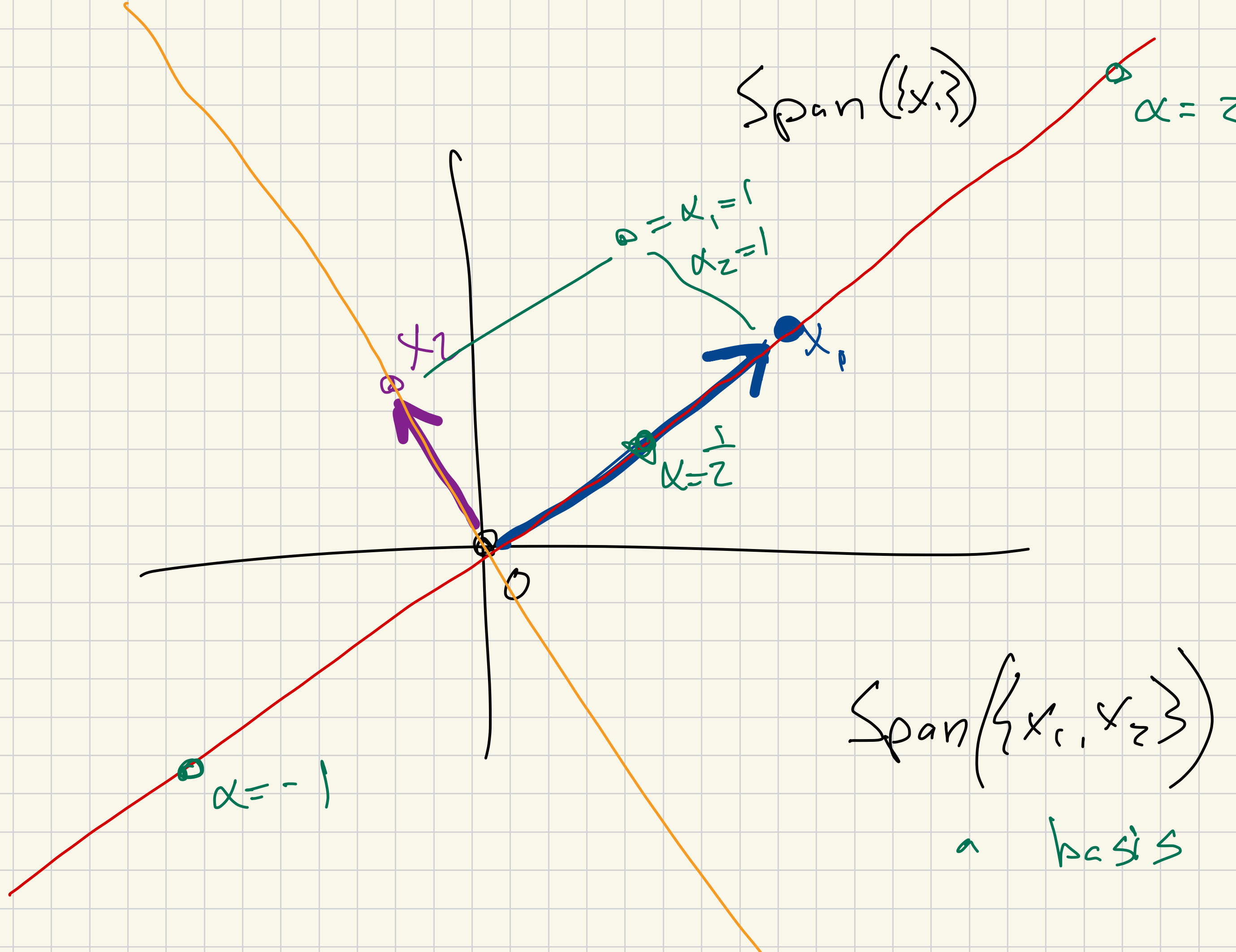
k scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$

$$z = \sum_{i=1}^{k} \alpha_i x_i \in \mathbb{R}^d \qquad \text{for choices of } \alpha_1 \alpha_2 \ldots \alpha_k$$

$$\text{span}(X) = \left\{ z \;\middle|\; z = \sum_{i=1}^{k} \alpha_i x_i, \ \alpha_i \in \mathbb{R} \right\}$$

such that

if $\text{span}(X) = \mathbb{R}^d \implies X$ is <u>basis</u>



$\mathrm{Span}(\{x_1, x_2\}) = \mathbb{R}^2$

a basis

Set vectors $X = \{x_1, \dots x_k\} \subset \mathbb{R}^d$

$z \in \mathbb{R}^d$ is linear independent of X

if $\nexists$ $\alpha_1, \alpha_2 \dots \alpha_k$ so $z = \sum_{j=1}^{k} \alpha_j x_j$

there does not exist

z is linear indep (X) $z \notin Span(X)$

otherwise z is linearly dependent of X

if $z \in Span(X)$

set X linearly independent (implicitly of itself)

if for each $z = x_j \in X$

no $\alpha_1, \alpha_2 \dots \alpha_{j-1}, \alpha_{j+1}, \dots \alpha_k$ so

$$x_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i x_i$$

takes role of z

if span decrease $\mathrm{Span}(X) \neq \mathrm{Span}(X \setminus x_j)$

for any x_j

then linearly independent.

set $X = \{x_1, x_2\}$ $\qquad x_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^3$

$$z_1 = \overset{\alpha_1}{1}\, x_1 \overset{\alpha_2}{-2}\, x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$$

z_1 is lin dependent of X

$$z_2 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} = \underset{1}{\alpha_1}\, x_1 + \underset{1}{\alpha_2}\, x_2 = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \neq z_2$$

z_2 is lin independent of X