L8: Linear Algebra Review #1

vectors, matrices, and multiplication

oh my!

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FoDA

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Vector

$v = (v_1, v_2, \dots, v_d) \in \mathbb{R}^d$

\in

$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix}$ column

$v^T = [v_1, v_2, \dots v_d]$

row

data point in $\mathbb{R}^d$

Matrix

A $n \times d$ $A \in \mathbb{R}^{n \times d}$

n rows d columns

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{array}{c} \\ n \text{ rows} \end{array} \begin{array}{c} \xrightarrow{d \text{ columns}} \\ \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & & & \\ A_{n1} & & \dots & A_{nd} \end{bmatrix} \end{array}$$

$\in \mathbb{R}^d$

n data points in $\mathbb{R}^d$

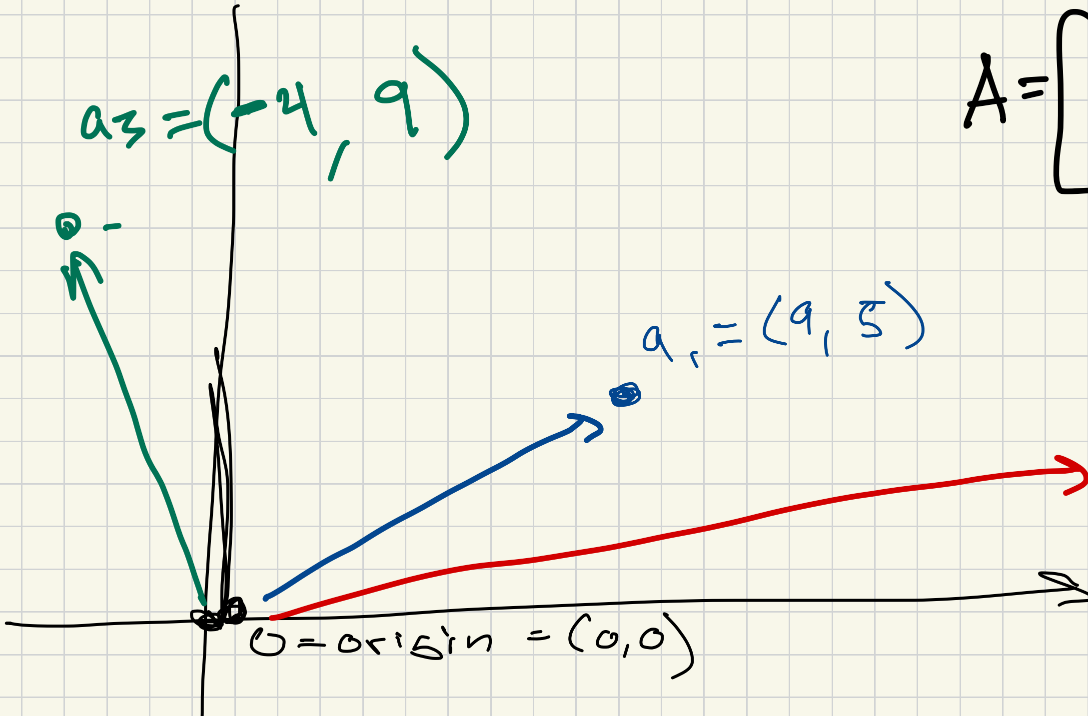
Vectors

$d = 2$

$\mathbb{R}^2$

$\vec{a_2} = (a_2 - O)$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 21 & 3 \\ -4 & 9 \end{bmatrix}$$



transpose $\quad (\cdot)^T$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n \times d} \Rightarrow A^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{d \times n}$$

Linear Equations

$n = 2$ equations

$d = \#$ variables $= 3$

$$\begin{array}{l} 3x_1 - 7x_2 + 2x_3 = -2 \\ -1x_1 + 2x_2 - 5x_3 = 6 \end{array} \qquad b = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \qquad A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$Ax = b$$

Addition

Subtraction ?

$$x = (x_1, x_2 \ldots x_d) \in \mathbb{R}^d$$

$$y = (y_1\ y_2 \ldots y_d) \in \mathbb{R}^d$$

$$z = x \overset{-}{+} y = (x_1 \overset{-}{+} y_1,\ x_2 \overset{-}{+} y_2,\ \ldots\ x_d \overset{-}{+} y_d)$$

dimensions must match

$$C = A + B \qquad A, B \in \mathbb{R}^{n \times d}$$

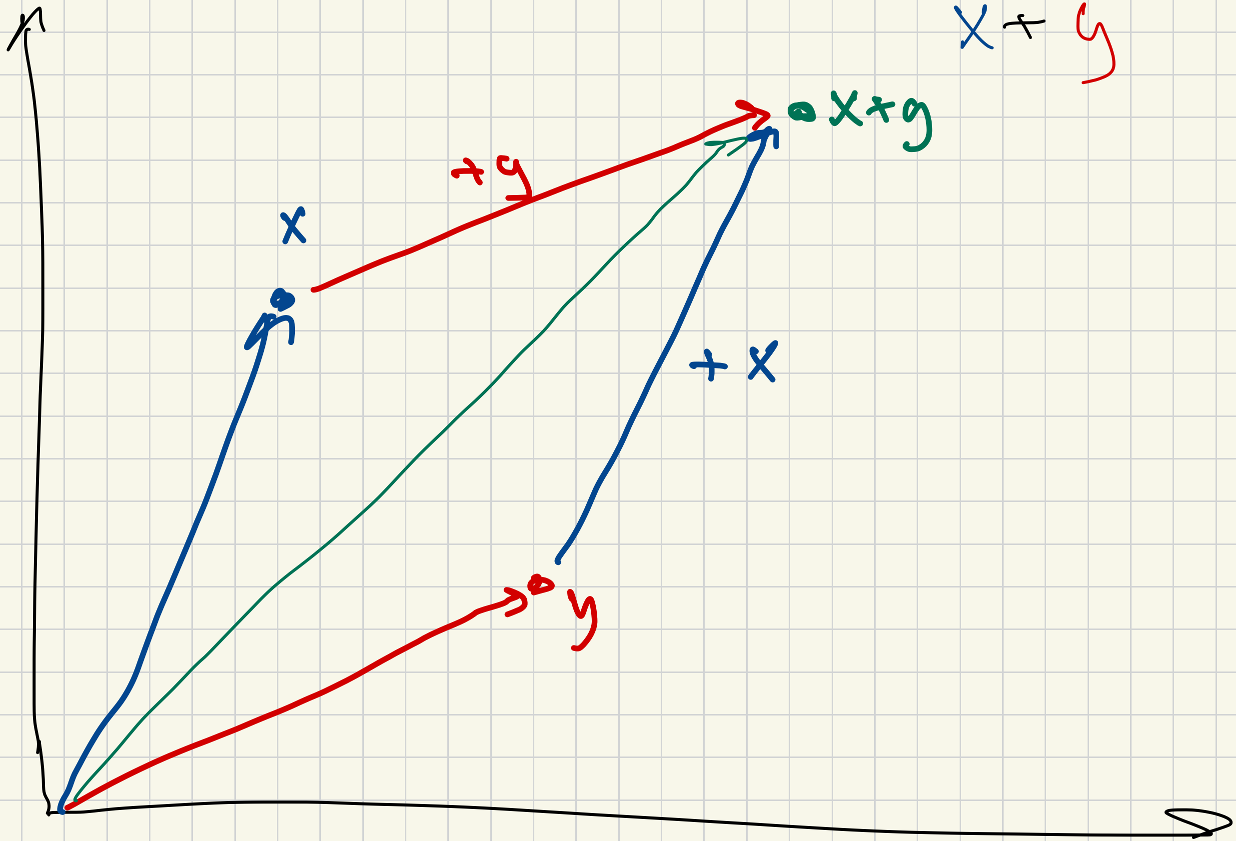
$$C_{ij} = A_{ij} + B_{ij}$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 8 & 1 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 10 & -4 \end{bmatrix}$$

$x + y$

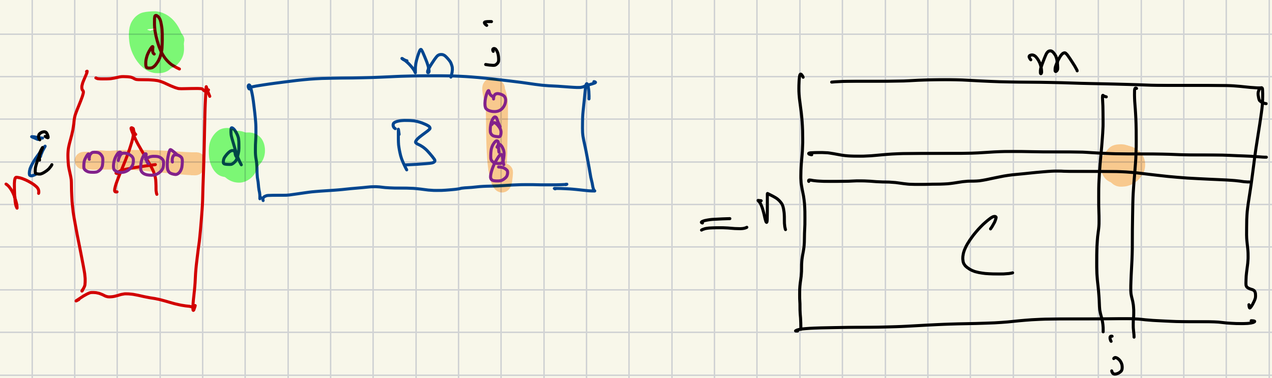


Multiplication

Matrix - Matrix Product

$A \in \mathbb{R}^{n \times d}$ $\qquad$ $B \in \mathbb{R}^{d \times m}$

$C = AB$ $\qquad$ $C_{ij} = \sum_{k=1}^{d} A_{ik} \cdot B_{kj}$



Matrix Multiplication

- associative $(AB)C = A(BC)$
- distributive $A(B+C) = AB + AC$
- <u>not</u> commutative $AB \neq BA$

$A \in \mathbb{R}^{n \times d}$ $B \in \mathbb{R}^{d \times m}$ AB legal

so if $n \neq m$ BA <u>not</u> legal.

Scalar - matrix product

$\alpha \in \mathbb{R}$ matrix $A \in \mathbb{R}^{n \times d}$

$$\alpha A = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1d} \\ & & & \\ & & & \alpha A_{nd} \end{bmatrix}$$

Vector-vector product

column $x, y \in \mathbb{R}^d$

inner product, dot product

$x \cdot y$

$$x^T y = \underset{1 \times d}{[x_1, x_2, \dots x_d]} \underset{d \times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{bmatrix}} = \sum_{i=1}^{d} x_i y_i = \langle x, y \rangle$$

\langle x , y \rangle

- associative
  $\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle x, \alpha y \rangle = \langle \sqrt{\alpha} x, \sqrt{\alpha} y \rangle$ $\quad \alpha > 0$
- distributive
  $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- commutative $\quad \langle x, y \rangle = \langle y, x \rangle$

$$? \; \left\langle \langle x, y \rangle, z \right\rangle \quad \text{legal?}$$

only if $z \in \mathbb{R}$ is <u>scalar</u>

$$\langle x, y \rangle \in \mathbb{R}$$

$u, v \in \mathbb{R}^2$

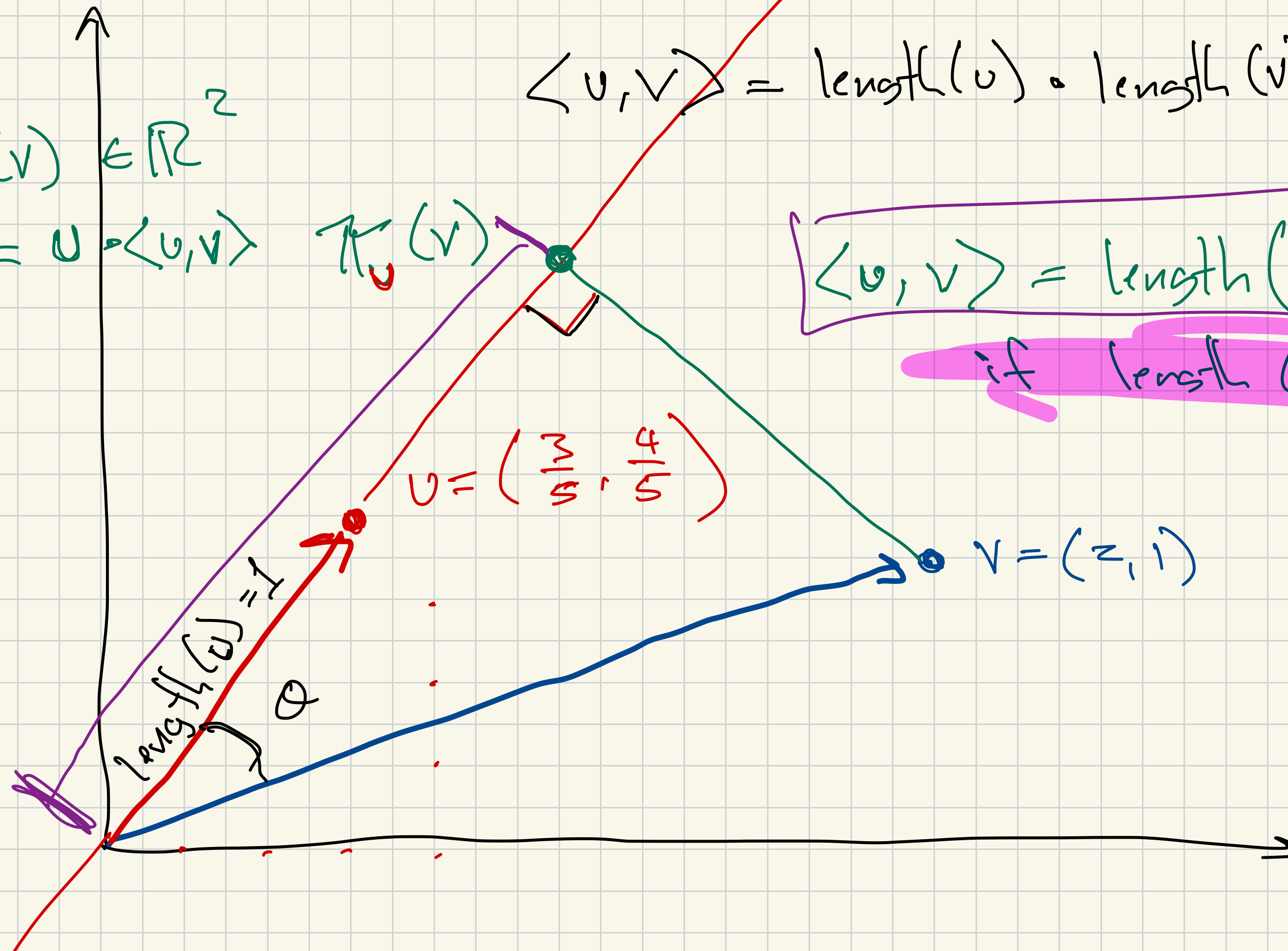
$\langle u, v \rangle = \text{length}(u) \cdot \text{length}(v) \cdot \cos(\theta)$

$\pi_u(v) \in \mathbb{R}^2$

$= u \cdot \langle u, v \rangle$

$$\boxed{\langle u, v \rangle = \text{length}(\pi_u(v))}$$

if $\text{length}(u) = 1$



outer - product

column $x \in \mathbb{R}^{n}$ $\quad y \in \mathbb{R}^{d}$

do not match.

$$x y^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_d \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & & & \\ \vdots & & & \\ x_n y_1 & & \cdots & x_n y_d \end{bmatrix} \in \mathbb{R}^{n \times d}$$

$n \times 1$ $\quad 1 \times d$

Matrix - vector product

$A \in \mathbb{R}^{n \times d}$ vector $x \in \mathbb{R}^{d}$

$y = Ax \in \mathbb{R}^{n}$

$$y = Ax = \begin{bmatrix} \boxed{x^T} \\ a_1 \\ a_2 \\ \\ a_n \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_n, x \rangle \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \\ y_n \end{bmatrix} \in \mathbb{R}^{n}$$