L7: Convergence

PAC & Concentration Bounds

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FoDA

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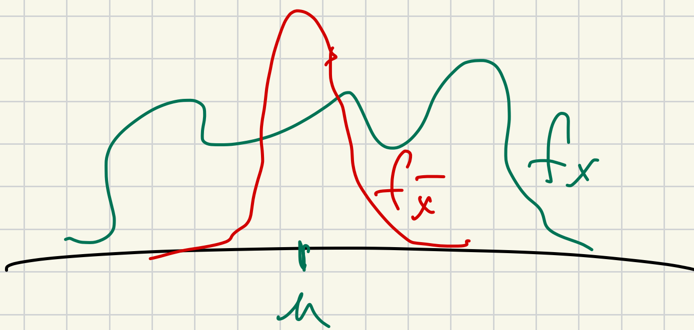
Review Central Limit Theorem

n RVs $X_1, X_2, \ldots X_n \overset{iid}{\sim} f_X$

- $E[X_i] = \mu$ - $Var[X_i] = \sigma^2 < \infty$

$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ analyze the average

- $\bar{X}$ converges to Normal
- $E[\bar{X}] = \mu$
- $Var[\bar{X}] = \frac{\sigma^2}{n}$



PAC

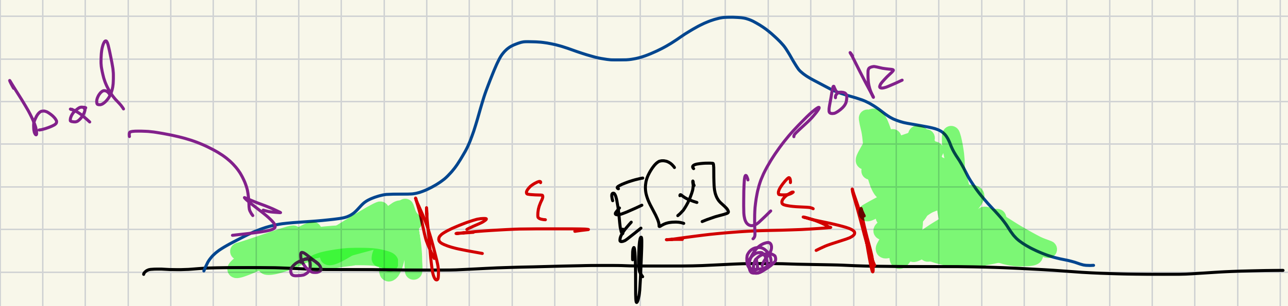
How accurate is $\bar{X}$?

close to $E[x]$

$$\Pr\left[\,|\bar{X} - E[x]| > \varepsilon\,\right] \le \delta$$

ε: error tolerance

δ: probability of failure



3 Convergence Inequalities

• Markov	$E[X]$, $X > 0$	1 RV
• Chebyshev	$E[X]$, $Var[X]$	1 RV or n RVs
• Chernoff – Hoeffding	$E[X]$, $X \in [a, b]$	n RVs

most powerful

average $\bar{X}$

Markov Inequality

RV X

Know

- $E[X]$
- $X \geq 0$

for any $\alpha > 0$

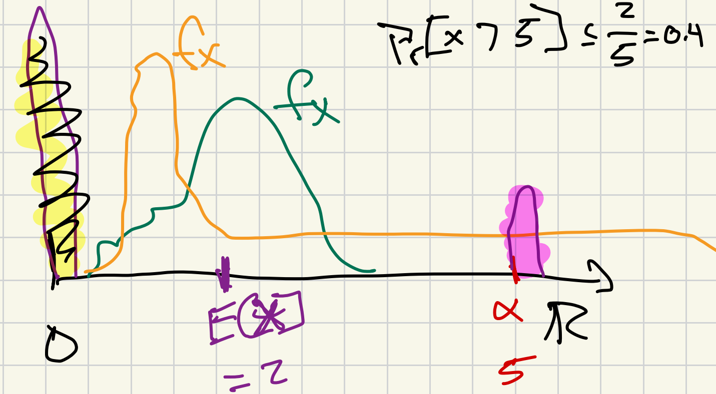
$$\Pr[X > \alpha] \leq \frac{E[X]}{\alpha} = \delta$$

$$E[X] = 0 \cdot \Pr[X=0] + \alpha \cdot \Pr[X=\alpha]$$

$$= 0 + \alpha \cdot \delta$$

Solve for $\frac{E[X]}{\alpha} = \delta$

$\Pr[X > 5] \leq \frac{2}{5} = 0.4$



PAC

$$\Pr[|X - E[X]| > \varepsilon] < \delta$$

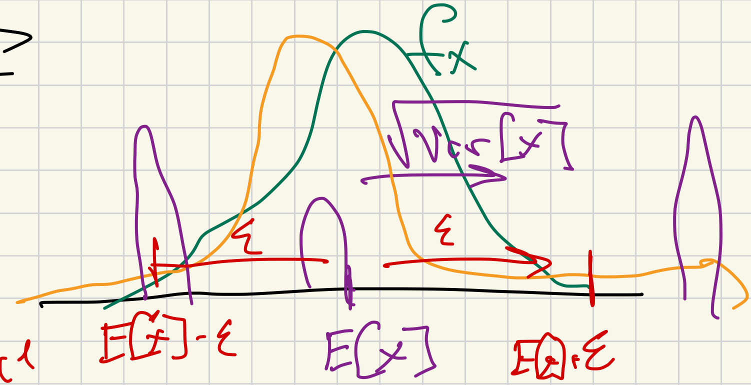
$\alpha = \varepsilon - E[X]$

$$\Pr[X - E[X] > \varepsilon - E[X]] \leq \boxed{\frac{E[X]}{\varepsilon - E[X]}} \to \delta$$

Chebyshev Inequality

RV X

Know
- $E[X]$
- $Var[X]$



error tolerance

$$\Pr\left[\,|X - E[X]| > \varepsilon\,\right] \le \frac{Var[X]}{\varepsilon^2} = \delta$$

RV S = snow in PC in Jan

- $S \geq 0$
- $E[S] = 20$ inches

$$\Pr[S \geq 50\text{ in}] \leq \frac{E[S]}{50\text{ in}} = \frac{20\text{ in}}{50\text{ in}} = 0.4$$

Markov

$\Pr[S \geq 50\text{in}]$

$\leq \min\{0.4, 0.01\}$

- $Var[S] = 9\text{ in}^2$

Chebyshev

$$\Pr[S \geq 50\text{ in}] = \Pr[S - E[S] \geq \underbrace{50\text{in} - \overset{20\text{in}}{E[S]}}_{30\text{in}}]$$

$$\leq \Pr[|S - E[S]| \geq 30\text{in}] \leq \frac{Var[S]}{(30\text{in})^2} = \frac{9\text{in}^2}{900\text{ in}^2} = \frac{1}{100}$$

Chebyshev PAC

n RVs $X_1, X_2, \ldots X_n$ $\quad \bar{X} = \frac{1}{n} \sum_i X_i$

- $E[\bar{X}] = E[X_i] = \mu$
- $Var[\bar{X}] = \frac{Var[X_i]}{n} = \frac{\sigma^2}{n}$

$$\Pr[|\bar{X} - E[\bar{X}]| > \varepsilon] \leq \frac{Var[\bar{X}]}{\varepsilon^2} = \frac{Var[X]}{n\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

Chernoff - Hoeffing inequalites

RVs $X_1, X_2, \ldots X_n \underset{iid}{\sim} f_X$ $\quad \bar{X} = \frac{1}{n} \sum X_i$

Know

- $E[X_i] = \mu$
- $X_i \in [a, b]$ $\quad \Delta = b - a$

$\exp(z) = e^z$

$$\Pr[|\bar{X} - E[\bar{X}]| > \varepsilon] \leq 2\exp\left(\frac{-2\varepsilon^2 n}{\Delta^2}\right) = \delta$$

Solve for n

$$n = \frac{\Delta^2}{2\varepsilon^2} \ln\left(\frac{2}{\delta}\right)$$

Chebyshev solve for n

$$n = \frac{Var[X]}{\varepsilon^2 \delta}$$

Die = $\{1 \dots 6\}$ $\qquad n = 120$ rolls $\qquad \bar{T} = T/n$

RV $\quad T_i = \begin{cases} 1 & \text{if } 3 \\ 0 & \text{o.w.} \end{cases}$ $\qquad T = \sum_{i=1}^{n} T_i \equiv \#3\text{s}.$

$E[T] = 20$

$\Delta = 1$

$\bar{T} \in [0, 1]$

$$\Pr\left[\frac{T}{n} \underset{-\frac{1}{6}}{\geq} \frac{40}{120} \underset{-\frac{1}{6}}{}\right] \leq \Pr\left[\left|\bar{T} - E[\bar{T}]\right| \geq \frac{1}{6}\right]$$

① $\frac{1}{n}$

② $- E[\bar{T}]$

③ $\leq$ via $|\cdot|$

$$\leq 2\exp\left(-\frac{2\left(\frac{1}{6}\right)^2 \cdot \overset{n}{120}}{1^2}\right)$$

$$\leq 2\exp\left(-\frac{20}{3}\right) \leq 0{,}0026$$

Chebyshev $\quad Var[T_i] = 5/36$

$$\Pr[T > 40] \leq \Pr\left[\left|\bar{T} - E[\bar{T}]\right| \geq \frac{1}{6}\right] \leq \frac{Var[T]}{n \cdot \left(\frac{1}{6}\right)^2} = \frac{5/36}{120 \cdot \left(\frac{1}{6}\right)^2} \leq 0.42$$