L19: Eigendecomp + Power Method

1. Best Rank-k Apx
2. Eigen-decomp $\Longleftrightarrow$ SVD
3. Power Method Algo.

Mar 23, 2026

FoDA

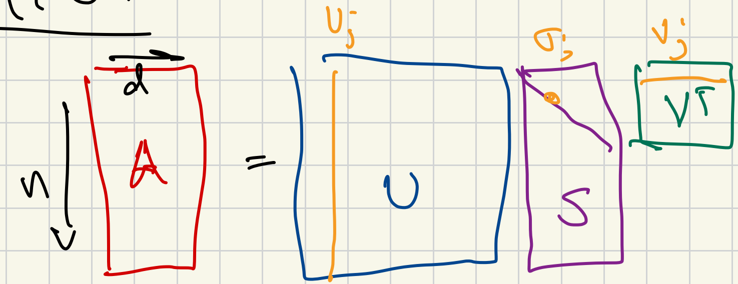
Jeff M. Phillips

Rank-k Approximation

Input $A \in \mathbb{R}^{n \times d}$ $\quad n > d$

$U, S, V^T \quad \text{svd}(A)$

$A = USV^T$



Goal: $A' \in \mathbb{R}^{n \times d} \quad \text{rank}(A') \leq k \quad k < d < n$

$$A_k = \underset{\text{rank}(A') = k}{\text{argmin}} \; \| A - A_k \|_{\text{For } 2}$$

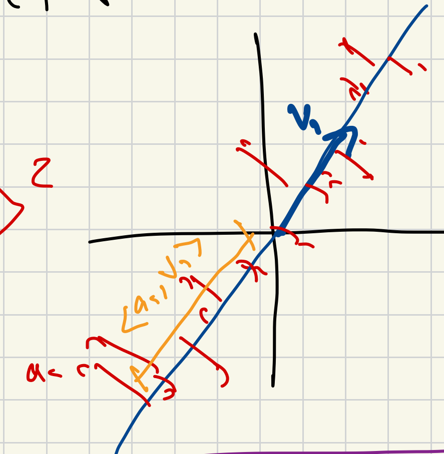
$$A_k = \sum_{j=1}^{k} \sigma_j \boxed{u_j v_j^T} \in \mathbb{R}^{n \times d}, \text{ rank } 1$$

$$A - A_k = \sum_{j=k+1}^{d} \sigma_j u_j v_j^T \longleftarrow \text{tail}$$

$$A = \sum_{j=1}^{d} \sigma_j u_j v_j^T$$

v_1 = first right singular vector

$$= \underset{\substack{v \in \mathbb{R}^d \\ \|v\| = 1}}{\arg\max} \underbrace{\| \boxed{A v} \|_2}_{\hookrightarrow \sum_{i=1}^{n} \langle a_i, v \rangle^2} \quad (Av \in \mathbb{R}^n)$$



σ_1 = first singular value

$$= \| A v_1 \| = \sqrt{\sum_{i=1}^{n} \langle a_i, v_1 \rangle^2}$$

$$= \| A \|_2 = \max_{\substack{v \in \mathbb{R}^d \\ \|v\| = 1}} \| A v \|$$

σ_j^2 = jth sing value sqrd

$$= \| A \underbrace{v_j}_{\hookrightarrow \text{jth right sing vec.}} \|^2$$

V_2 = 2nd right sing vector

unit vector v maximizes $\|Av\|$ → hard to interpret by itself.

but $\langle v, v_1 \rangle = 0$

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T \in \mathbb{R}^{n \times d}$$

first k singular values

$$\underbrace{A_k}_{n \times d} = \underbrace{U_k}_{n \times k} \; \underbrace{S_k}_{k \times k} \; \underbrace{V_k^T}_{k \times d}$$

$v_1, \ldots, v_k$ first k right sing vector

$u_1 \;\; u_k$

first k Left sing. vectors

Square Matrices

$M \in \mathbb{R}^{d \times d}$

positive definite

$\hookrightarrow \text{rank}(M) = d$

eigen value $\lambda_j \in \mathbb{R}$

eigen vector $v_j \in \mathbb{R}^d$

$$M v_j = \lambda_j v_j \qquad \|v_j\| = 1$$

---

eigen decomposition

$$M = V L V^T$$

a set d $\{(v_1, \lambda_1), (v_2, \lambda_2) \ldots (v_d, \lambda_d)\}$

$\lambda_j \geq \lambda_{j+1} \qquad \langle v_j, v_{j'} \rangle = 0 \quad j \neq j'$

$V = [v_1 \ v_2 \ \ldots \ v_d] \in \mathbb{R}^{d \times d} \qquad L = \text{diag}(\lambda_1, \ldots \lambda_d) \in \mathbb{R}^{d \times d}$

$$M_R = A^T A \in \mathbb{R}^{d \times d} \qquad A \in \mathbb{R}^{n \times d}$$

↳ if A has $n > d$, full rank
↳ M_R p.d.

| if not full rank → M_R p.s.d.

$$A = USV^T \Leftarrow \mathrm{svd}(A)$$

$I = U^{-1}U$ $\quad$ $I = V^{-1}V$

$$\boxed{M_R V =} A^T A V = (V S^T U^T)(U S V^T)\, V$$

$$= V S^T S = \boxed{V S^2}$$

$S^2 = \mathrm{diag}(\sigma_1^2, \sigma_2^2 \dots \sigma_d^2)$

one column

$$M_R v_j = v_j \sigma_j^2 \Longrightarrow$$

v_j eigenvector of M_R

σ_j^2 eigenvalue of M_R

$\lambda_j = \sigma_j^2$

(sing values)^2 = eigenvalues

right svectors = eigenvectors.

$$M_L = AA^T \in \mathbb{R}^{n\times n}$$

$$u_j \; M_L = \sigma_j^2 u_j$$

left sing vectors u_j of A

eigenvectors of $M_L = AA^T$

still eigenvalues M_L $\lambda_j = \sigma_j^2$

if $n > d$ what is λ_j for $j > d$.

$\hookrightarrow \lambda_j = 0 \quad j > d.$

How to compute inverse

$M_R \in \mathbb{R}^{d \times d}$ $p:d$

$M_R = A^T A$

Linear regression

$\alpha^* = (A^T A)^{-1} A^T y$

$$M_R^{-1} = (V L V^T)^{-1} = V^{-1} L^{-1} V$$

$$= V^T L^{-1} V$$

$$L = \mathrm{diag}(\lambda_1, \ldots \lambda_d)$$

$$L^{-1} = \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_d}\right)$$

$$M_R^{-1} = V^T \cdot \mathrm{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_d}\right) V$$

Power Method (to compute eigendecomp)

Input $M \in \mathbb{R}^{d \times d}$ # iterations $q \approx 10-20$

0. $v^{(0)} \in \mathbb{R}^d$, $\|v^{(0)}\| = 1$, at random

guess at v_1 ← top eigenvector

1. for $i = 1$ to q

$$v^{(i)} = M v^{(i-1)}$$

2. Return $v_1 = \frac{v^{(q)}}{\|v^{(q)}\|}$

for $i = 1$ to q

$v = Mv$

q = # steps

larger if $\frac{\lambda_1}{\lambda_2}$ small.

$v_1 \leftarrow \text{Power}(M)$

$x = M v_1$

$M^{(1)} = M - x x^T$

$v_2 \leftarrow \text{Power}(M^{(1)})$

factor out v_2, repeat

$$M = \sum_{j=1}^{d_1} v_j \lambda_j v_j^T \quad = V L V^T$$

$$v = \left(M \dots \left(M \left(M \left(M v^{(0)}\right)\right)\right) \dots\right)$$

$$= M^q v^{(0)}$$

$$M^2 = \left(V L \boxed{V^T}\right)\left(\boxed{V} L V^T\right)$$

I

$$= V L^2 V^T$$

$$M^q = V L^q V^T$$

$\lambda_1 = 10 \quad \lambda_1^2 = 100 \quad \lambda_1^3 = 1000$

$\lambda_2 = 2 \quad \lambda_2^2 = 4 \quad \lambda_2^3 = 8$

$$\lim_{q \to \infty} V \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} V^T$$