L18: Singular Value Decomposition

(the SVD)

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FoDA

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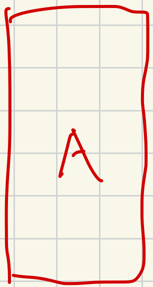
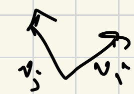
Input: $A = \{a_1, a_2, \dots, a_n\} \subset \mathbb{R}^d$ $\quad A \in \mathbb{R}^{n \times d}$



Goal: Basis B $\quad V_B = \{v_1, v_2 \dots v_k\}$ $\quad k \le d$

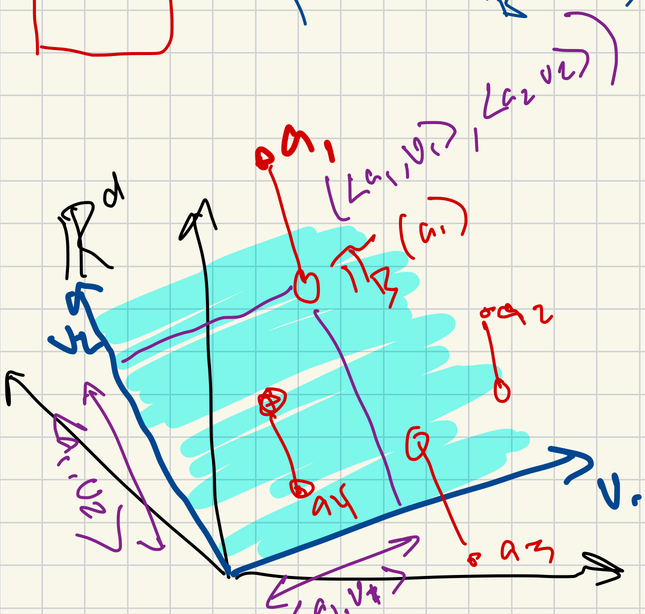
$\|v_j\| = 1 \quad \langle v_j, v_i \rangle = 0$



projection $\pi_B(a)$

Choose V_B

$$V_B^* = \{v_1, \dots v_k\} = \arg\min_{V_B} \sum_{i=1}^{n} \|\underbrace{a_i - \pi_B(a_i)}_{\text{residual}}\|^2$$



either

$\rightarrow \pi_B(a_i) \in \mathbb{R}^d$

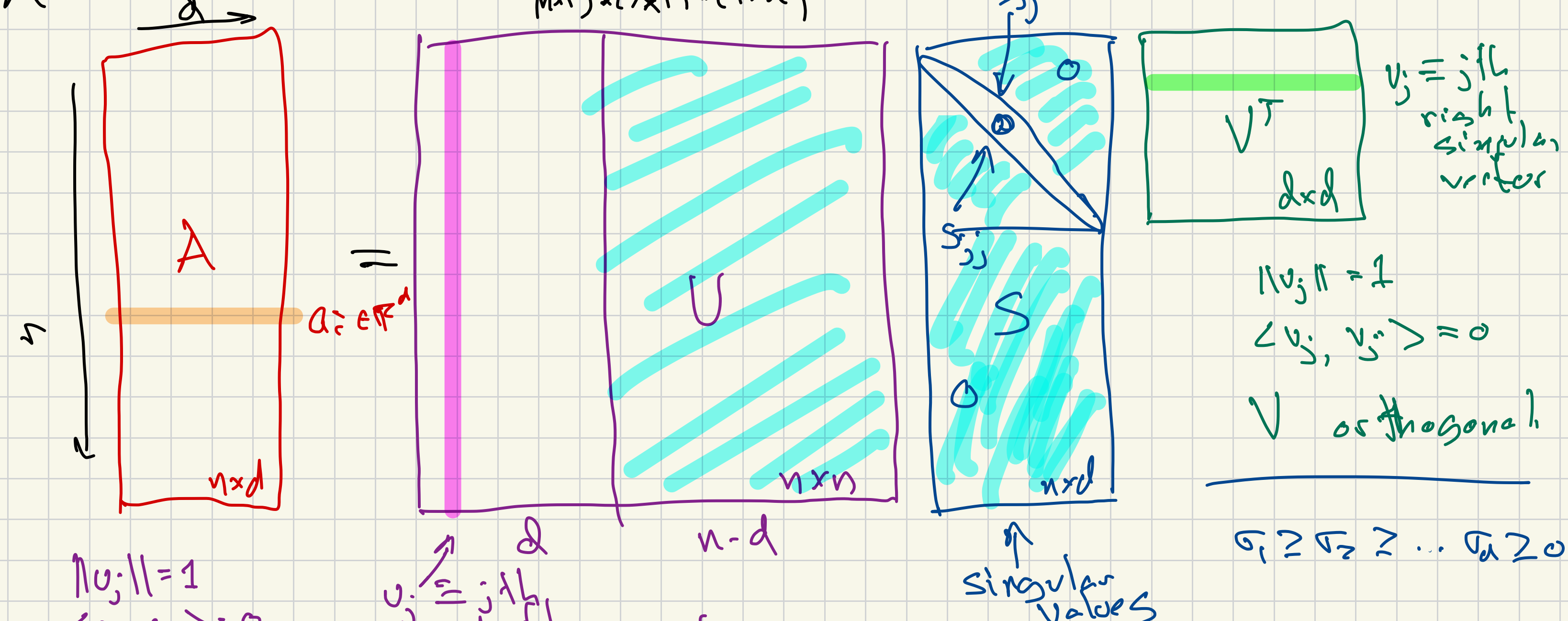
$\rightarrow (\langle a_i, v_1 \rangle, \langle a_i, v_2 \rangle \dots \langle a_i, v_k \rangle) \in \mathbb{R}^k$

Singular Value Decomposition (SVD)

$A = USV^T$

$A = \sum_{j=1}^{d} \underbrace{u_j \sigma_j v_j^T}_{(n\times 1)\times(1\times 1)\times(1\times d)} \in \mathbb{R}^{n\times d}$

$S_{jj} = \sigma_j$ = jth singular value



$\|u_j\| = 1$

$\langle u_j, u_{j'} \rangle = 0$

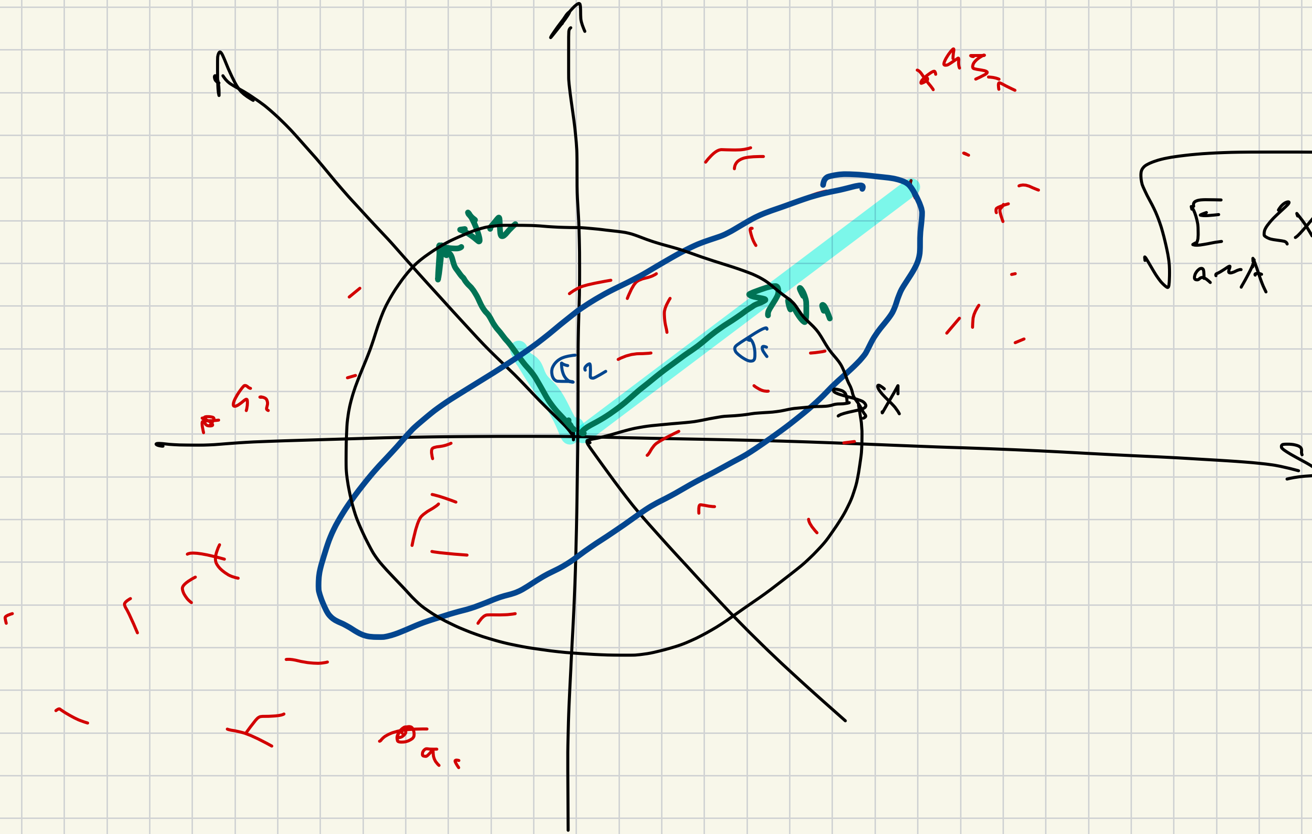
U orthogonal

$\|v_j\| = 1$

$\langle v_j, v_{j'} \rangle = 0$

V orthogonal

$\sigma_1 \geq \sigma_2 \geq \dots \sigma_d \geq 0$

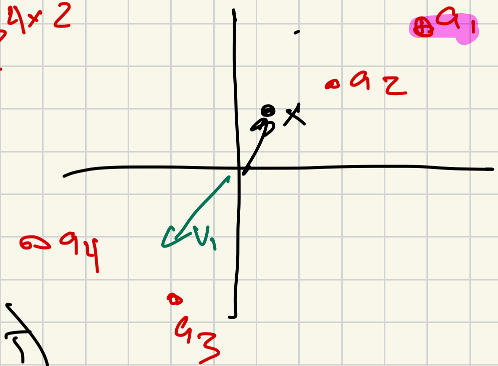


$$\sqrt{\sum_{a \sim A} \langle x, a_i \rangle^2}$$

svd(A)

$$V = \begin{array}{cc} v_1 & v_2 \\ \end{array}$$
$$V = \begin{pmatrix} -0.8142 & -0.5805 \\ -0.5805 & 0.8142 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix} \in \mathbb{R}^{4\times 2}$$



$$S = \begin{pmatrix} 8.1655 & 0 \\ 0 & 2.3074 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

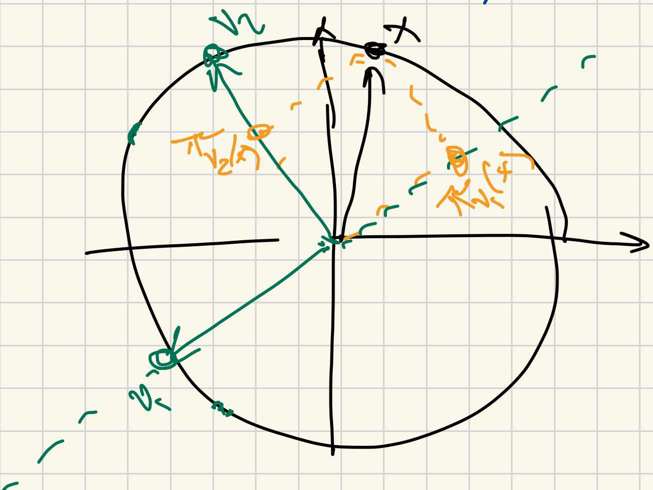
$$Ax = (USV^T)x$$

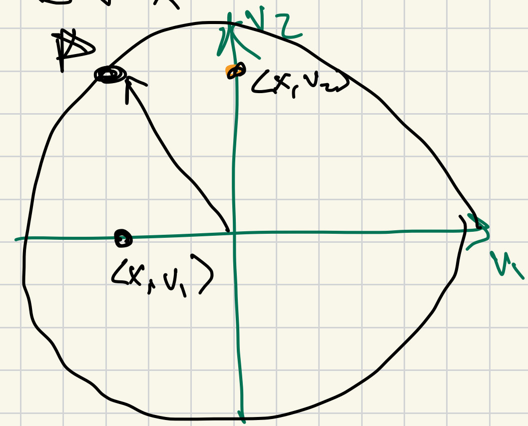
new coordinate frame $\longrightarrow V^T x$

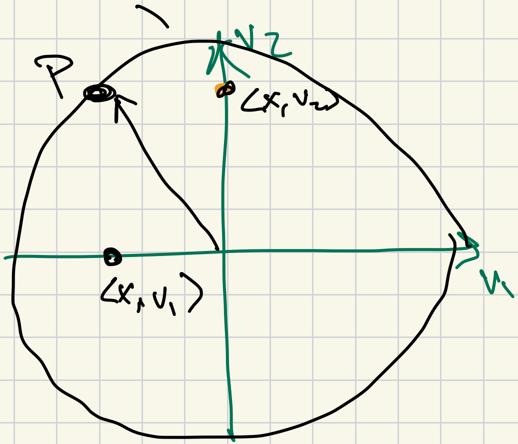
$x = (0.243, 0.970)$

$SV^T x$

$p = V^T x = (-.5, .9)$





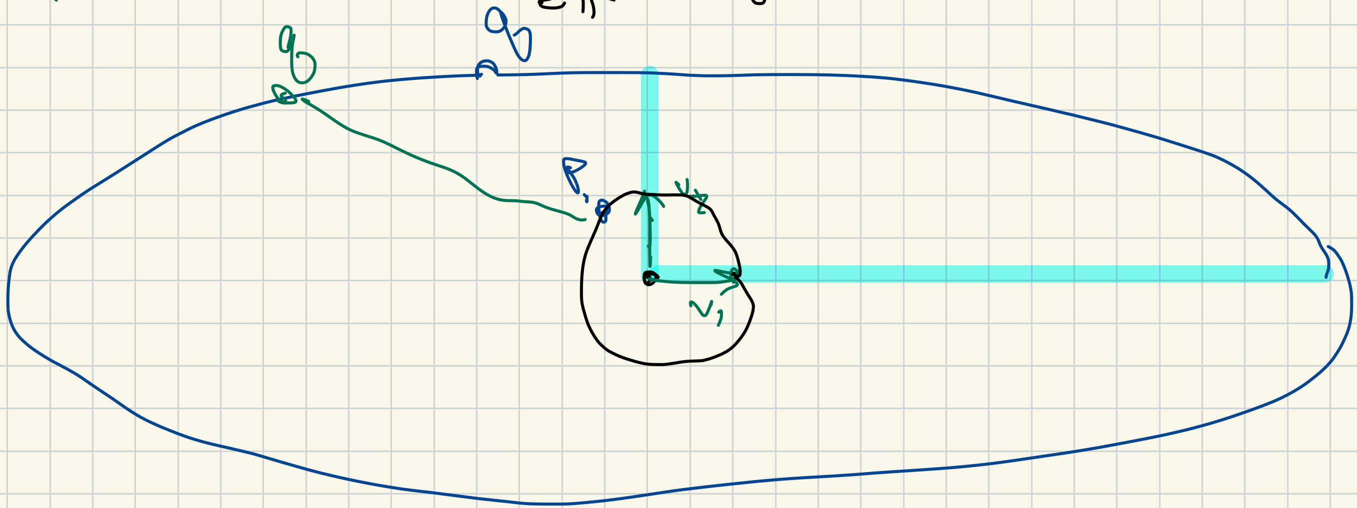


$$p = V^T x = \left(\langle v_1, x \rangle , \langle v_2, x \rangle \right)$$

$$q = S V^T x = S p$$

$$S = \begin{pmatrix} 8.16 & 0 \\ 0 & 2.3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\in \mathbb{R}^4$ $\quad q = (q_1, q_2, 0, 0)$



Input $A \in \mathbb{R}^{n \times d}$

$$\mathrm{svd}(A) = U S V^T \qquad V = \{v_1, v_2 \ldots v_d\}$$

fixed k

$$\{v_1, v_2, \ldots v_k\} = \underset{\substack{V_B \\ |V_B| = k}}{\mathrm{argmin}} \sum_{i=1}^{n} \| a_i - \pi_B(a_i) \|^2$$

(given B contains origin)

What is $\displaystyle\sum_{i=1}^{n} \| a_i - \pi_{B^*}(a_i) \|^2$

$$= \sum_{j=k+1}^{d} \sigma_j^2$$

bottom d-k sing values squared.

$$\sum_{i=1}^{n} \|a_i - \Pi_B(a_i)\|^2$$

$V_B = \{v_1, v_2 \ldots v_k\} \leftarrow$ top k right SV.

$$= \sum_i \Big\| \sum_{j=1}^{d} \underbrace{v_j \langle v_j, a_i \rangle}_{\Pi_{v_j}(a_i)} - \sum_{j=1}^{k} v_j \langle v_j, a_i \rangle \Big\|^2$$

$$= \sum_i \Big\| \sum_{j=k+1}^{d} v_j \langle v_j, a_i \rangle \Big\|^2$$

pythagorian

$\langle v_j, v_{j'} \rangle = 0$

$$= \sum_i \sum_{j=k+1}^{d} \| v_j \langle v_j, a_i \rangle \|^2$$

$\|v_j\| = 1$

$$= \boxed{\sum_i} \sum_{j=k+1}^{d} \boxed{\cancel{\|v_j\|^2}^{\,1} \langle v_j, a_i \rangle^2} = \sum_{j=k+1}^{d} \|A v_j\|^2 = \sum_{j=k+1}^{d} \sigma_j^2$$

Best Rank-k Approximation

$$A \in \mathbb{R}^{n\times d} \Longrightarrow A_k \in \mathbb{R}^{n\times d} \qquad \mathrm{rank}(A_k) = k$$

"best"

$$A_k = \underset{\mathrm{rank}(B)=k}{\mathrm{argmin}} \boxed{\|B-A\|_F^2} \quad \text{or} \quad \underset{\mathrm{rank}(B)=k}{\mathrm{argmin}} \boxed{\|B-A\|_2^2}$$

$= \sum_{i=1}^{n} \|\pi_B(a_i) - a_i\|^2 = \sum_{j=k+1} \sigma_j^2$

$= \sigma_{k+1}^2$

both $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$

top k singular vectors + values u_j v_j σ_j

$\|B-A\|_2^2 = \max_{x} \sum_i \langle x, b_i - a_i \rangle^2$

$\|x\| = 1$