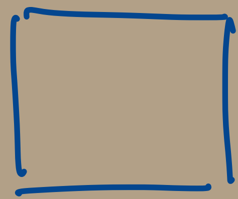
L10: Linear Algebra Review pt.3

Square Matrices



Feb 9, 2026

F.DA

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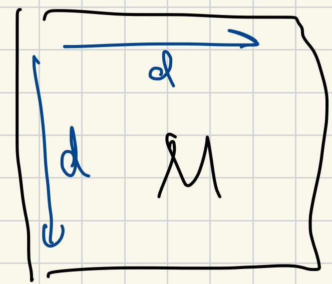
Square Matrices $\quad M \in \mathbb{R}^{d\times d}$



prior: $+$, $-$, $\times$ (dot product)

Inverse: division $\div$

scalar α: $\alpha^{-1} = \frac{1}{\alpha}$

matrix $M \in \mathbb{R}^{d\times d}$ $\qquad \alpha \cdot \alpha^{-1} = \alpha \cdot \frac{1}{\alpha} = 1$

$$\underbrace{(M^{-1})}_{\text{inverse of } M} M = I = M(M^{-1})$$

$$I = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{d\times d}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cannot always invert M.

M invertible if $\begin{cases} 1.\ \text{square.} \\ 2.\ \text{full rank.} = \operatorname{rank}(M) = d. \end{cases}$

Eigenvalues & Eigenvectors

square matrix $M \in \mathbb{R}^{d \times d}$

$$M v = \lambda v$$

eigenvector $v \in \mathbb{R}^d$

eigenvalue $\lambda \in \mathbb{R}$

$\|v\| = 1$

$\{(v_1, \lambda_1), (v_2, \lambda_2), \dots (v_r, \lambda_r)\}$ $\quad r = \text{rank}(M)$

Positive Definite Matrix

$M \in \mathbb{R}^{d \times d}$ s.t. d eigenvectors, values

real.

each $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$

Data Matrix $A \in \mathbb{R}^{n \times d}$ row $a_i \in \mathbb{R}^d$ 1 data point.

$n > d$, rank$(A) = d$,

$A^T A = M \in \mathbb{R}^{d \times d} \Rightarrow M$ pd. = (positive definite)

Positive Semidefinite Matrix $M \in \mathbb{R}^{d \times d}$ (psd.)

d eigenval, vect

each $\lambda_i \in \mathbb{R}$ $\lambda_i \geq 0$

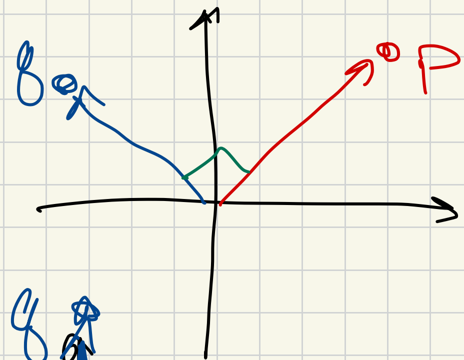
$\Leftarrow$ A not full rank

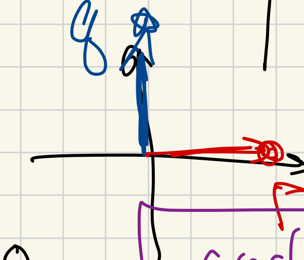
$M = A^T A$

Orthogonality

Vectors $p, q \in \mathbb{R}^d$

orthogonal iff $\langle p, q\rangle = 0$





$p = (1, 0)$ $\quad$ $q = (0, 1)$

$\langle p, q\rangle = 1\cdot 0 + 0\cdot 1 = 0$

$p = (1, 1)$ $\quad$ $q = (-1, 1)$

$\langle p, q\rangle = 1\cdot(-1) + (1)\cdot(1)$

$-1 + 1 = 0$

ang(1 $\theta_{p,q}$

$\langle p, q\rangle = \|p\|\cdot\|q\|\cdot\cos\theta$

$\theta = \arccos\left(\frac{\langle p, q\rangle}{\|p\|\cdot\|q\|}\right)$

$p = (2, -3, 4, -1, 6)$ $\quad$ $q = (4, 5, 3, -7, -2)$

$\langle p, q\rangle = \underbrace{2\cdot 4}_{8} + \underbrace{(-3)5}_{-15} + \underbrace{4\cdot 3}_{12} + \underbrace{(-1)\cdot(-7)}_{7} + \underbrace{6(-2)}_{-12} = 0$

matrix $V \in \mathbb{R}^{n \times d}$ $\quad$ d columns $v_1, v_2, \ldots v_d$

set $\{v_j\}$ = orthogonal $\quad$ if $\langle v_i, v_j \rangle = 0$

all pair v_i, v_j orthogonal.

set $\{v_j\}$, **orthonormal** if

1. orthogonal
2. all v_j $\quad \|v_j\| = 1$ (normalized)

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I \qquad i=j \qquad i \neq j$$

matrix V **orthogonal** if $\quad (V^T V)_{ij} = \langle v_i, v_j \rangle$

$$V^T V = I$$

$= 0$ unless $i = j$

= all column vect $\{v_j\}$ orthonormal.

then $\langle v_i, v_j \rangle = 1$

Orthogonal Matrix

$V \in \mathbb{R}^{d \times d}$

- square
- all columns orthogonal, normalized

$$V^T V = I$$

- also all rows orthogonal, normalized
  - ↳ orthonormal matrix

$$V^T V = I = V V^T$$

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orthogonal V define $V^{-1} = V^T$

View orthogonal matrix as a rotation

vector $x \in \mathbb{R}^d$ ortho matrix $U \in \mathbb{R}^{d\times d}$

$$\|Ux\| = \|x\|$$