FoDA L8

Linear Algebra Review

Vectors, Matrices, & Multiplication

Sep 15, 2022

Vector $\quad v = (v_1, v_2, \dots, v_d) \in \mathbb{R}^d$

in

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} \qquad v^T = \begin{bmatrix} v_1 & v_2 & \dots & v_d \end{bmatrix}$$

column $\qquad$ row

$\leftarrow$ data point in $\mathbb{R}^d$

Matrix $\quad A \quad n \times d \quad A \in \mathbb{R}^{n \times d}$

n rows $\quad d$ columns

$$A = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_n - \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nd} \end{bmatrix}$$

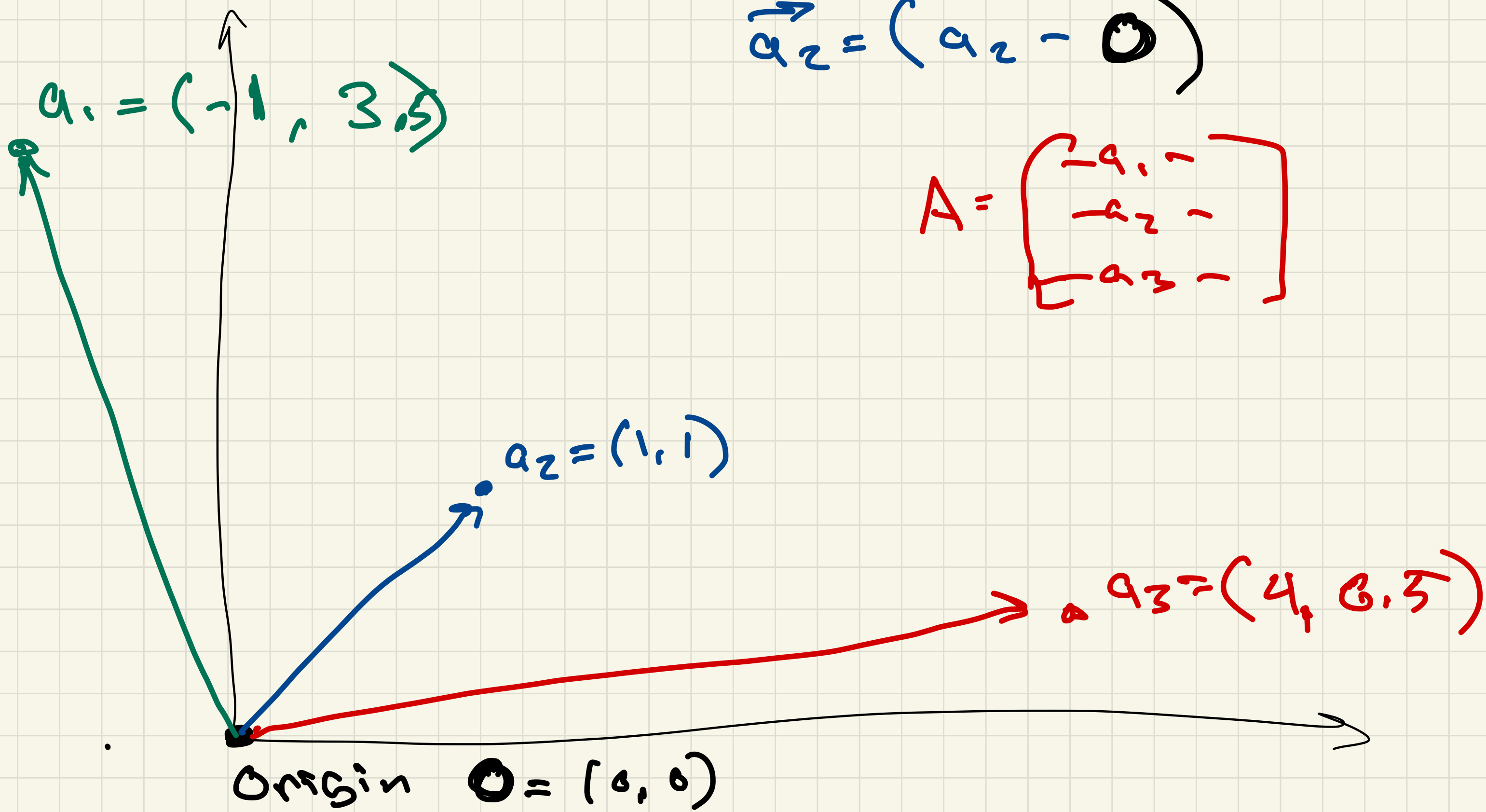
$\leftarrow$ n data points in $\mathbb{R}^d$

Vectors

$d = 2$ $\mathbb{R}^2$

$$\vec{a_2} = (a_2 - O)$$

$$A = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ - & a_3 & - \end{bmatrix}$$



transpose $(\cdot)^T$

$$A = \begin{bmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_n & - \end{bmatrix} \in \mathbb{R}^{n \times d} \implies A^T = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{d \times n}$$

Linear Equations

$$3x_1 - 7x_2 + 2x_3 = -2$$
$$-1x_1 + 2x_2 - 5x_3 = 6$$

n = equations

d = # variables

A

x

b

$$Ax = b$$

$$b = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \in \mathbb{R}^2 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

Addition

$$x = (x_1, x_2 \dots x_d) \in \mathbb{R}^d$$
$$y = (y_1, y_2 \dots y_d)$$

$$z = x \mp y =$$
$$(x_1 \mp y_1, \; x_2 \mp y_2, \; \dots, \; x_d \mp y_d) \in \mathbb{R}^d$$

dimension must match

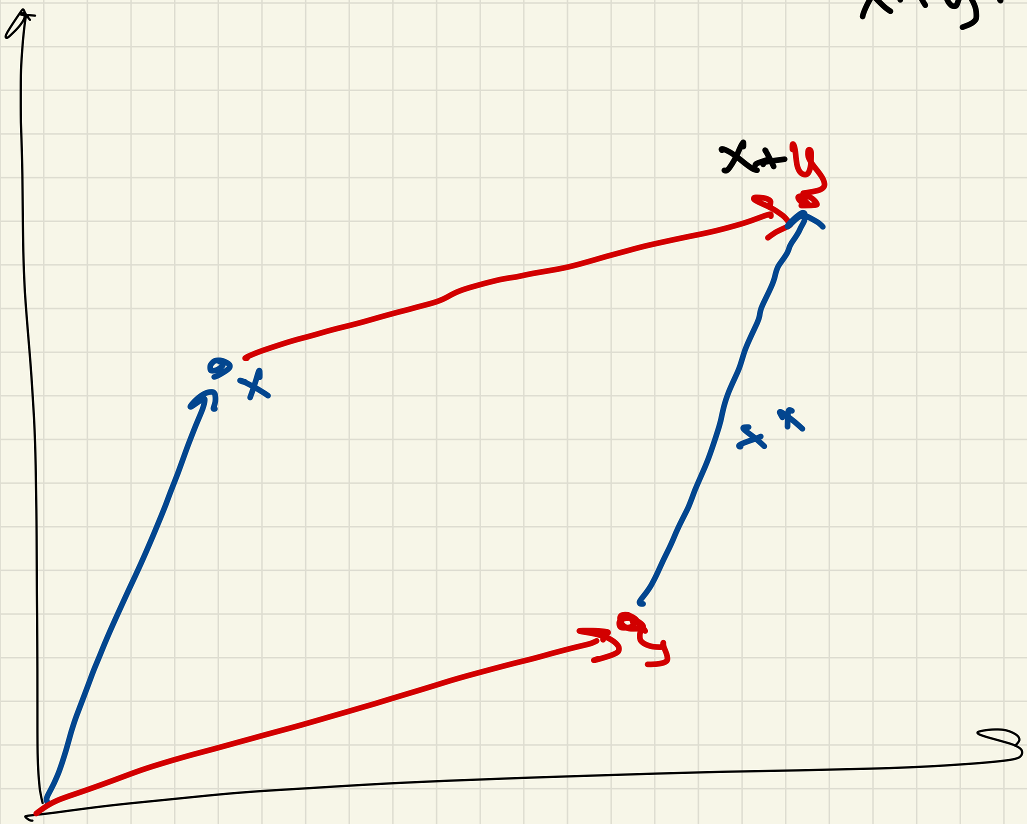
$$A, \; B \in \mathbb{R}^{n \times d}$$

$$C = A + B \qquad C_{ij} = A_{ij} + B_{ij}$$

$$A = \begin{bmatrix} 3 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 8 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & -2 & 5 \\ 3 & 10 & -4 \end{bmatrix}$$

$$x + y = y + x$$

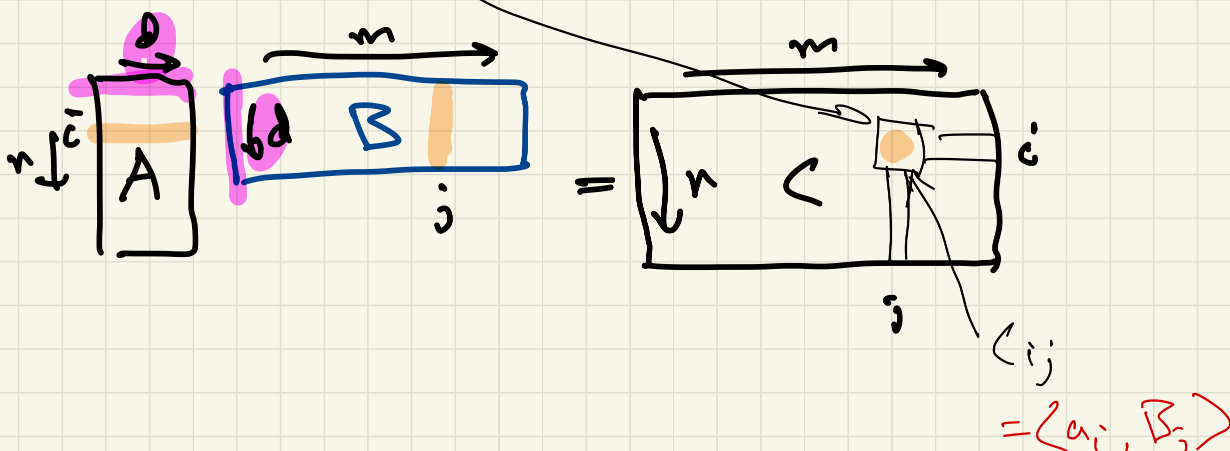


Matrix-Matrix Product

$A \in \mathbb{R}^{n \times d}$ $\quad$ $B \in \mathbb{R}^{d \times m}$

$C = AB$ $\quad$ $C_{i,j} = \sum_{k=1}^{d} A_{i,k} \cdot B_{k,j}$

$\in \mathbb{R}^{n \times m}$



Matrix Multiplication

- associative $(AB)C = A(BC)$
- destributive $A(B+C) = AB + AC$
- <u>not</u> commutative $AB \neq BA$

$A \in \mathbb{R}^{n \times d}$ $B \in \mathbb{R}^{d \times m}$

$n \neq d \neq m$
$n \neq m$

AB

BA not legal operation

scalar - matrix product

scalar $\rightarrow \alpha \in \mathbb{R}$ matrix $A \in \mathbb{R}^{n \times d}$

$$\alpha A = \begin{bmatrix} \alpha A_{11} & & \\ & \ddots & \\ & & \alpha A_{n \times d} \end{bmatrix}$$

Vector-vector product

column $x, y \in \mathbb{R}^d$

inner product, dot product

$$x^T y = \begin{bmatrix} x_1 & \dots & x_d \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix} = \sum_{i=1}^{d} x_i y_i = \langle x, y \rangle$$

length x, y

angle

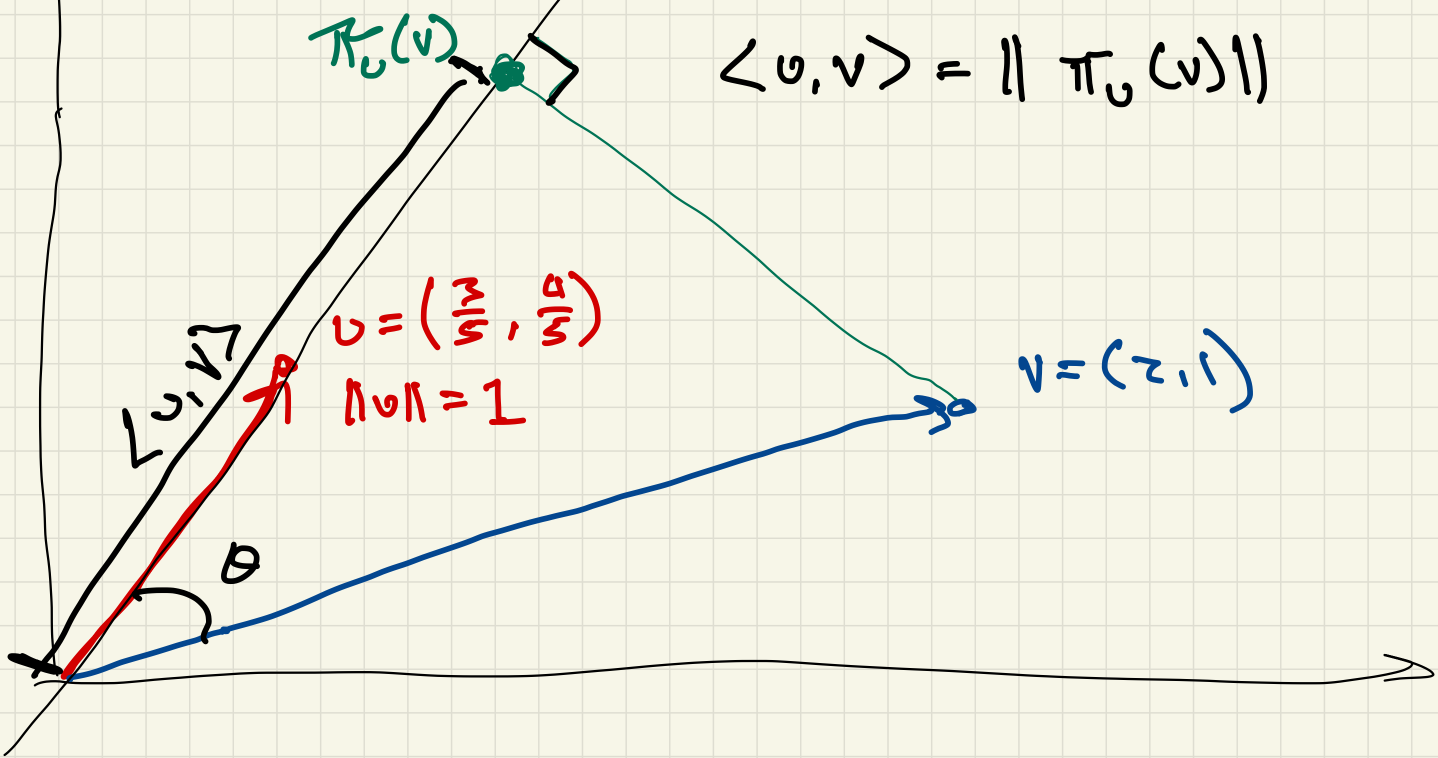
- associative
- distributive
- commutative $\langle x, y \rangle = \langle y, x \rangle$

$$\langle \alpha x, y + z \rangle = \alpha \langle x, y + z \rangle = \alpha \left(\langle x, y \rangle + \langle x, z \rangle \right)$$

$$\langle u, v\rangle = \text{length}(u) \cdot \text{length}(v) \cdot \cos(\theta_{u,v})$$
$$\qquad\qquad\quad \|u\| \cdot \|v\|$$

$$\pi_u(v) = u(\langle u, v\rangle)$$

$$\langle u, v\rangle = \|\pi_u(v)\|$$



outer product

column

$x \in \mathbb{R}^n \qquad y \in \mathbb{R}^d$

$$x y^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_d \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & & x_1 y_d \\ x_2 y_1 & & & \vdots \\ \vdots & & \ddots & \vdots \\ & & & x_n y_d \end{bmatrix} \in \mathbb{R}^{n \times d}$$

Matrix - Vector product

$A \in \mathbb{R}^{n \times d}$ vector $x \in \mathbb{R}^{d \times 1}$

$y = Ax \in \mathbb{R}^n$

$$y = Ax = \begin{bmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_m - \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} \langle a_1, x \rangle \\ \langle a_2, x \rangle \\ \vdots \\ \langle a_n, x \rangle \end{bmatrix} \in \mathbb{R}^n$$

$a_i \in \mathbb{R}^d$