FoDA L10

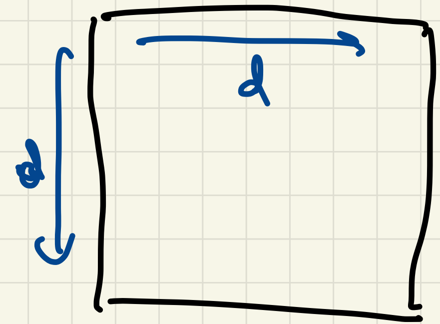
Linear Algebra Review #3

Square Matrices: Inverse, PSD, Orthogonality

Sep 22, 2022

Square Matrices

$M \in \mathbb{R}^{d \times d}$



Inverse M^{-1}

scalar α : $\alpha^{-1} = \frac{1}{\alpha}$

dividing

$$(M^{-1})M = I = MM^{-1}$$

$$\alpha \cdot \frac{1}{\alpha} = 1$$

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in \mathbb{R}^{d \times d}$$

Cannot always invert a matrix

if you can: invertable

- square
- full rank
  $\Rightarrow \text{rank}(M) = d$

Eigenvalue & Eigenvectors

square matrix $M \in \mathbb{R}^{d \times d}$

$$M v = \lambda v$$

eigenvector
$v \in \mathbb{R}^d$

eigenvalue
$\lambda \in \mathbb{R}$

usually $\|v\| = 1$

$r = \mathrm{rank}(M)$

$(v_1, \lambda_1), (v_2, \lambda_2), \ldots (v_r, \lambda_r)$

Positive Definite Matrices

$M \in \mathbb{R}^{d \times d}$

d eigenvectors, values

each $\lambda_i \in \mathbb{R}$ $\quad \lambda_i > 0$

$A \in \mathbb{R}^{n \times d}$ $\quad n > d$ $\quad \text{rank}(A) = d$

$A^T A = M \implies M$ pd (pos. def.)

Positive Semi Definite $M \in \mathbb{R}^{d \times d}$ (psd)

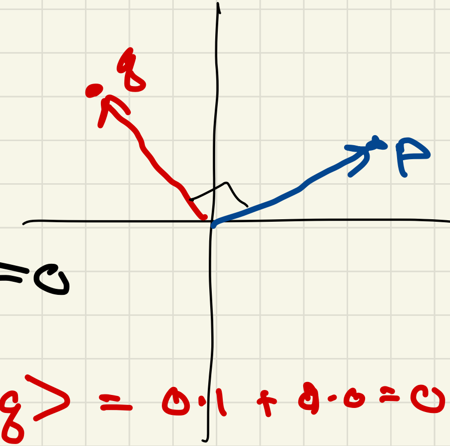
d eigen val, vec

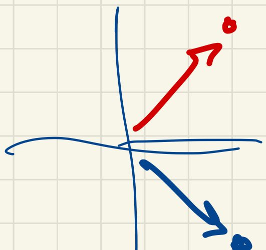
each $\lambda_i \in \mathbb{R}$ $\quad \lambda_i \geq 0$

Orthoganility

vectors $p, q \in \mathbb{R}^d$

orthogonal iff $\langle p, q \rangle = 0$





$p = (1,0)$ $\quad q = (0,1)$ $\quad \langle p, q \rangle = 0 \cdot 1 + 1 \cdot 0 = 0$

$p = (1,-1)$ $\quad q = (1,1)$ $\quad \langle p, q \rangle = 1 \cdot 1 + (-1) \cdot 1$

$= 1 - 1 = 0$

$p = (2, -3, 4, -1, 6)$ $\quad q = (4, 5, 3, -7, -2)$

$8 + (-15) + 12 + 7 + (-12) = 0$

$V \in \mathbb{R}^{n \times d}$ d columns $v_1, v_2, \ldots v_d$

if all v_i, v_j orthogonal, $\|v_i\| = 1$

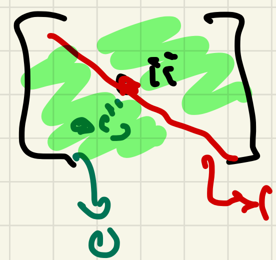
V orthonormal.

$V^T V = I$

$(V^T V)_{ij} = \langle v_i, v_j \rangle$

if $i \neq j \Rightarrow 0$

$i = j \Rightarrow \langle v_i, v_i \rangle = \|v_i\|^2$



$V V^T \neq I$

Orthogonal Matrix $U \in \mathbb{R}^{d \times d}$

- square
- all rows orthogonal, normalize

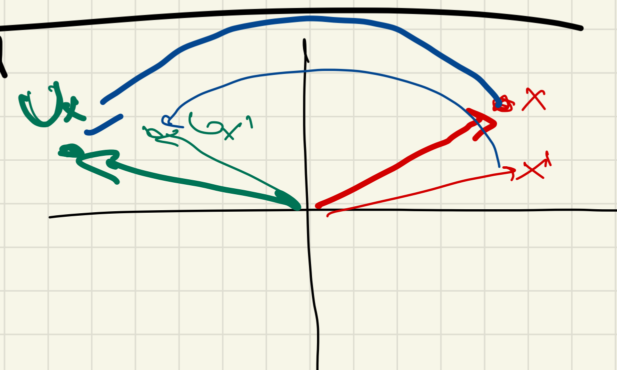
orthonormal

- all columns orthogonal, normalize

$$U^T U = I = U U^T$$

for any vector $x \in \mathbb{R}^d$

$\|x\| = \|Ux\|$



Columns $u_1, \dots u_d$ of U orthogonal matrix

orthogonal basis

$$\langle u_i, u_j \rangle = 0 \quad i \neq j$$

$$\|u_i\| = 1$$

$$Ux = \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_d, x \rangle \end{bmatrix}$$

$$x = \sum_{j=1}^{d} \alpha_j u_j$$