# L17: SVD and Relatives

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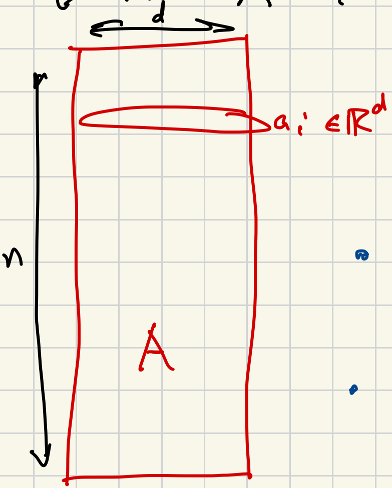
Data $A = \{a_1, a_2, \dots a_n\} \subset \mathbb{R}^d$ $A \in \mathbb{R}^{n \times d}$




## Dimensionality Reduction

- each column has same units
- learned representation "distributed"

Goal: Mapping $\mu: \mathbb{R}^d \to \mathbb{R}^k$
- $\mu(a) \in \mathbb{R}^d$ on k-dim subspace

or
- $\bar{\mu}(a) \in \mathbb{R}^k$

# Projection (Linear

$u = \pi_F(x)$

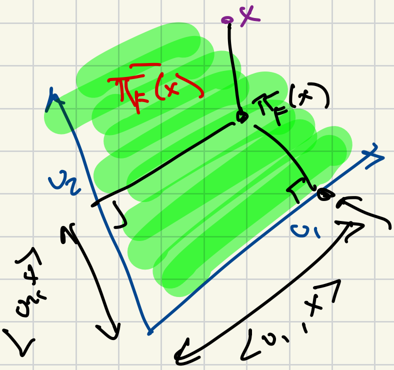
$F = \{v_1, v_2, \dots v_k\}$

$v_j \in \mathbb{R}^d$

$\|v_j\| = 1$


$\langle v_j, v_{j'} \rangle = 0$

$\bar{\pi}_F(x) = (\langle v_1, x\rangle, \langle v_2, x\rangle)$

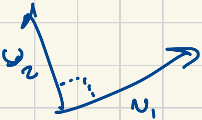



$v_1 \cdot \underline{\langle v_1, x\rangle}$

$\pi_F = v_1 \cdot \langle v_1, x\rangle + v_2 \langle v_2, x\rangle$

$\pi_{v_1}(x) + \pi_{v_2}(x)$

# Random Projections




d = 1 million

k = 500

Pick each $u_j$ at random

independent

$\langle u_j, u_{j'} \rangle \approx 0$

$\neq 0$

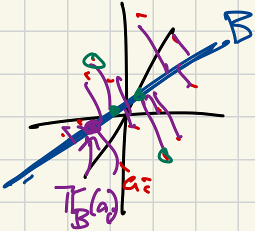
• Preserve <u>all</u> distances $(1 \pm \epsilon)$

$$(1-\epsilon)\|x-x'\| \le \underbrace{\|u(x)-u(x')\|}_{k\text{-dim}} \le (1+\epsilon)\|x-x'\|$$

$$k \approx \frac{1}{\epsilon^2}\log n$$

# Sum of Squared Errors $\quad SSE(A,B)$

$$SSE(A,B) = \sum_{a_i \in A} \| a_i - \underbrace{\pi_B(a_i)}\|^2$$




Goal Find k-dim subspace $B$

$$B^* = \arg\min_B SSE(A,B)$$

Key tool S.V.D (PCA)

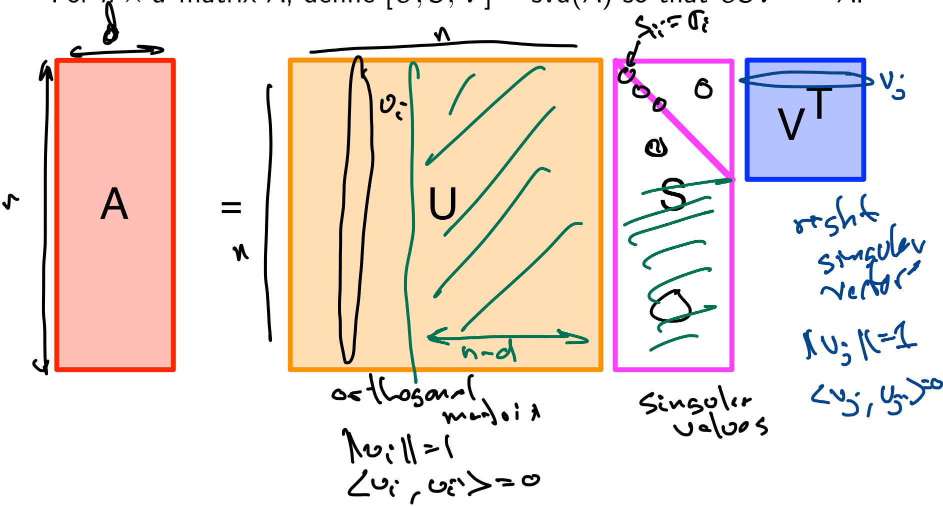
$\hookrightarrow$ Principal Component Analysis

# Singular Value Decomposition

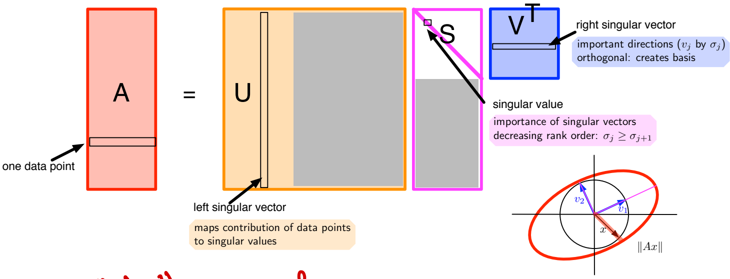
$x \in \mathbb{R}^d$

$V^T x \in \mathbb{R}^d$

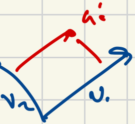
For $n \times d$ matrix $A$, define $[U, S, V] = \text{svd}(A)$ so that $USV^T = A$.



# Singular Value Decomposition




$\|Ax\|$ $\quad x \in \mathbb{R}^d$ $\quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_d$




B = subspace of low-rank approximate of A

$B = \{v_1, v_2, \dots v_k\} \leftarrow SVD(A)$

$$\sum_i \| a_i - \pi_B(a_i) \|^2$$

$$\sum_i \Big\| \sum_{j=1}^{d} v_j \langle v_j, a_i \rangle - \sum_{j=1}^{k} v_j \langle v_j, a_i \rangle \Big\|^2$$

$$\sum_i \Big\| \sum_{j=k+1}^{d} v_j \langle v_j, a_i \rangle \Big\|^2$$

pythagorean theorem

unit vectors

$$\sum_i \sum_{j=k+1}^{d} \| v_j \langle v_j, a_i \rangle \|^2$$

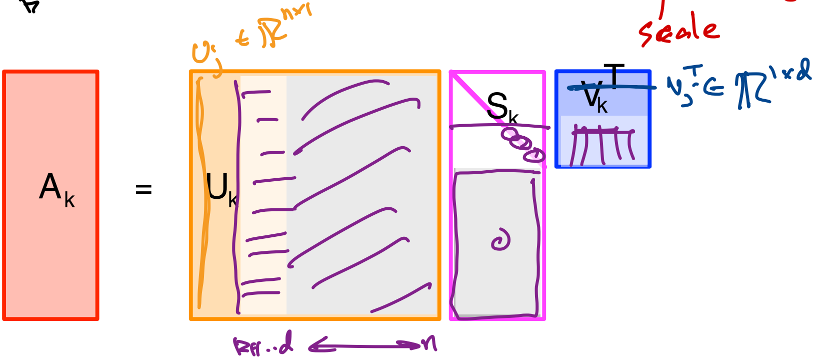
$$\sum_i \sum_{j=k+1}^{d} \langle v_j, a_i \rangle^2$$

$$\sum_{j=k+1}^{d} \boxed{\sum_i \langle v_j, a_i \rangle^2} = \sum_{j=k+1}^{d} \| A v_j \|^2 = \sum_{j=k+1}^{d} \sigma_j^2$$

# Best Rank $k$-Approximation

$$\underset{B \text{ rank } k}{\arg\min} \|A - B\|_{F,2} \Rightarrow B^* = A_k$$

$$A_k = \sum_{j=1}^{k} \underset{\substack{= \\ \text{scale}}}{\sigma_j} \underbrace{(u_j v_j^T)}_{\|u_j v_j^T\|_F = 1}$$




$$A_k = U_k S_k V_k^T$$
$$= \sum_{j=1}^{k} u_j \sigma_j v_j^T \in \mathbb{R}^{n \times d}$$

Want draw r-d picture $A \in \mathbb{R}^{n \times d}$

Step 1 $\quad SVD(A) = USV^T$

$\boxed{\text{Option A}}$ $\quad \forall a_i \rightarrow (\langle a_i, v_1 \rangle, \langle a_i, v_2 \rangle)$

$\boxed{\text{Option B}}$ $\quad A_k = U_k S_k V_k^T \qquad U_k = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_k \\ | & | & & | \end{bmatrix}$

$b_i \leftarrow \pi_F(a_i)$

$b_i = \cancel{(u_1(i), u_2(i))}$

$b_i = (\sigma_1 u_1(i), \sigma_2 u_2(i))$

Find optimal $F = \{f_1, f_2 \ldots f_k\}$ $f_j \in \mathbb{R}^d$

$\|f_j\| = 1$

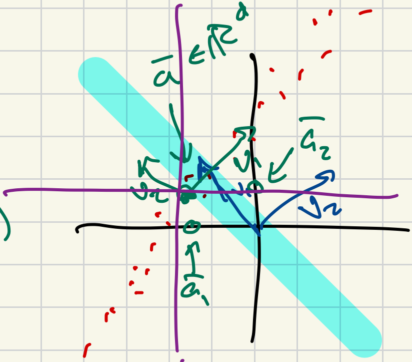
$$F^* = \underset{F}{\arg\min} \sum_{i=1}^{n} \| a_i - \pi_F(a_i) \|^2$$

PCA

Step 1 center $A$

$\bar{a} = (\bar{a}_1, \bar{a}_2 \ldots \bar{a}_d)$

$\bar{a}_j = \frac{1}{n} \sum_{i=1}^{d} A_{ij}$




$\bar{A} = \{ \bar{a}_i = a_i - \bar{a} \}$

Step 2 SVD $(\bar{A})$

centering

# Eigen Decomposition

$M \in \mathbb{R}^{d \times d}$ positive definite (positive semidefinite)

for any $x \in \mathbb{R}^d$

$x^T M x \underset{psd}{\geq} 0$

$M = A^T A$ for some $A \in \mathbb{R}^{n \times d}$

if $d < n$, A full rank

$\hookrightarrow$ pd.

eigenvector $v \in \mathbb{R}^d$ for $M \in \mathbb{R}^{d \times d}$ p.d.

$$M v = \overset{\in \mathbb{R}}{\lambda} v$$

eigenvector

eigenvalue

$d$ eigen vector/value pairs

$$(v_1, \lambda_1)\ (v_2, \lambda_2)\ \dots\ (v_d, \lambda_d)$$

$$\langle v_i, v_j \rangle = 0 \qquad \|v_j\| = 1$$

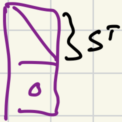
$$M = V L V^T$$

$$V = \begin{bmatrix} | & | & \\ v_1 & v_2 & \\ | & | & \end{bmatrix} \in \mathbb{R}^{d \times d} \qquad \mathrm{diag}(\lambda) = \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{bmatrix}}_{L}$$

start $\quad A \in \mathbb{R}^{n \times d} \quad A = \underbrace{U S V^T}$

$$M = A^T A = (V S^T U^T)(U S V^T)$$

$$= V S^T \underbrace{U^T U}_{I} S V^T$$




$$= \underbrace{V S^T} \underbrace{S V^T}$$

$$= V \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ 0 & & \ddots & \sigma_d^2 \end{bmatrix} V^T$$

right sing vectors
=
eigenvectors

$L \quad \lambda_j = \sigma_j^2$

eigenvalues = sq - sing values

$M = A A^T$ $\qquad A \in \mathbb{R}^{n \times d}$

$M \in \mathbb{R}^{n \times n}$

eigenvalues

$\text{eig}(M) \Rightarrow M = U \Lambda U^T$

left sing vectors