Geometry of Diverse, High-Dimensional, and Nonlinear Imaging Data

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Manifold Data in Vision and Imaging

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- Transformation groups (rotations, projective, affine)

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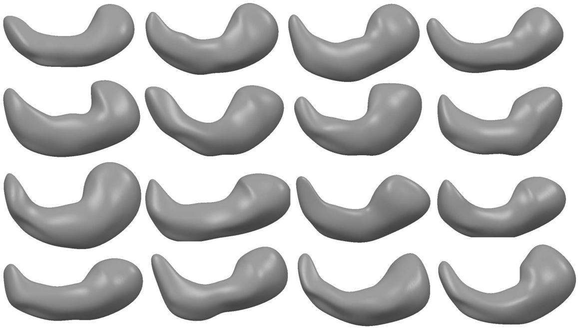
- Diffeomorphisms (deformable transformations)

Manifold Statistics: Averages



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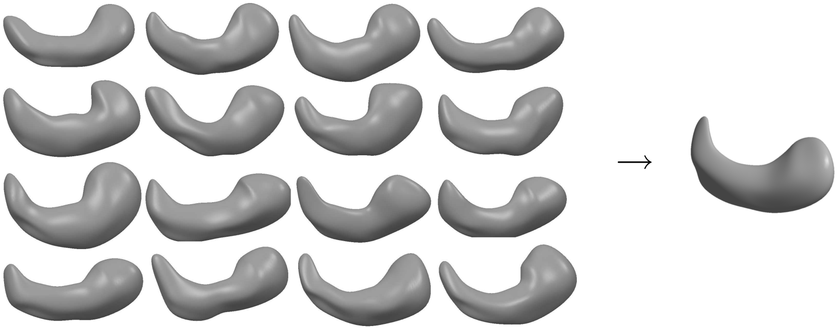
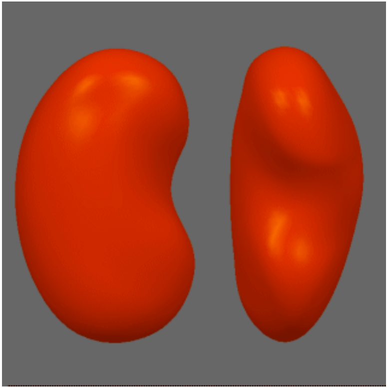
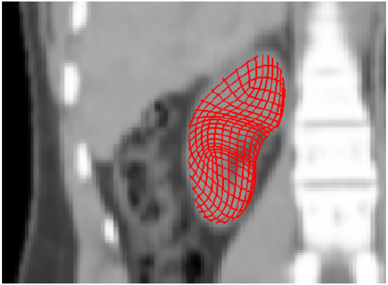


Manifold Statistics: Variability

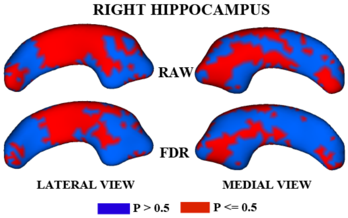


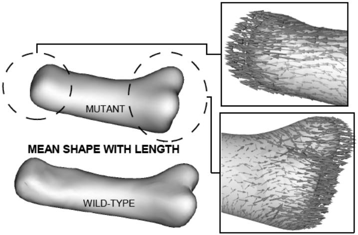


Shape priors in segmentation

Manifold Statistics: Hypothesis Testing

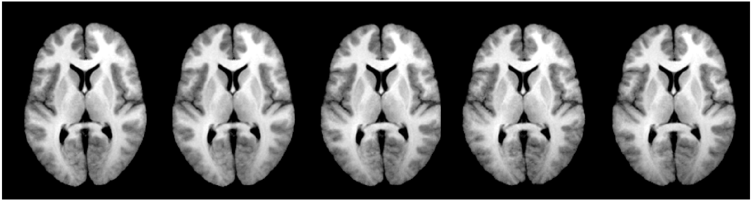
Testing group differences



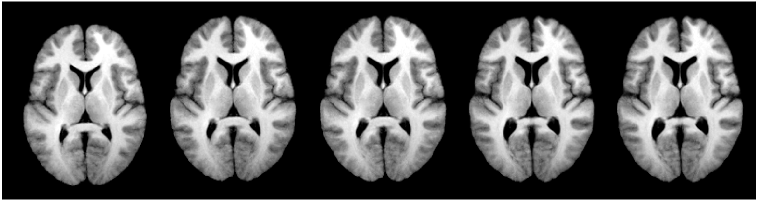


Cates, et al. IPMI 2007 and ISBI 2008.

Manifold Statistics: Regression

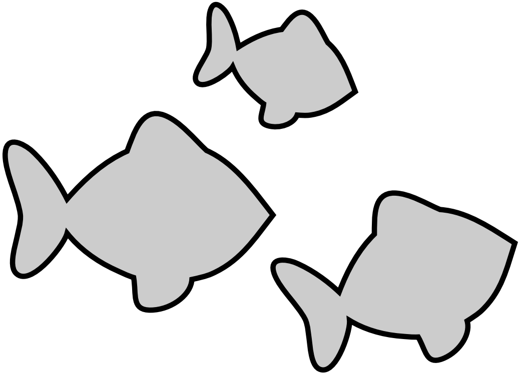


35 37 39 41 43



45 47 49 51 53

What is Shape?



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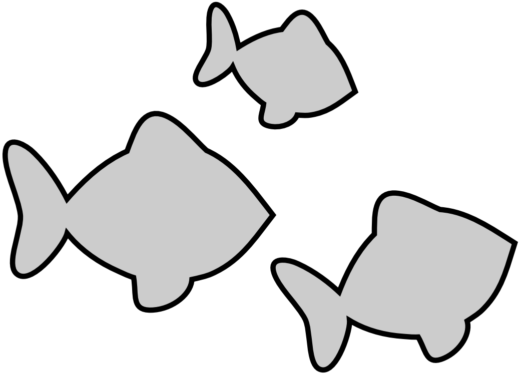
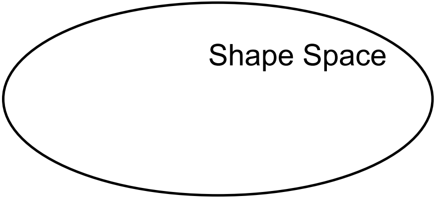


Shape is the geometry of an object modulo position, orientation, and size.

Shape Analysis



A shape is a point in a high-dimensional, nonlinear shape space.

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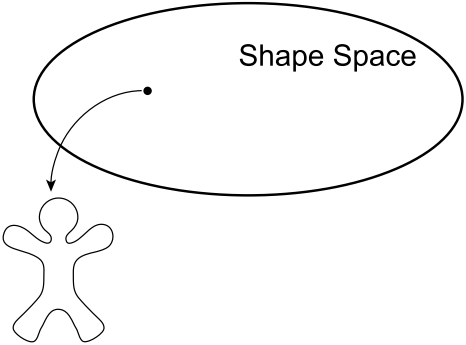

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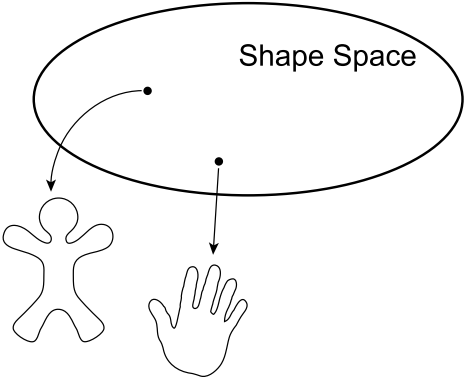

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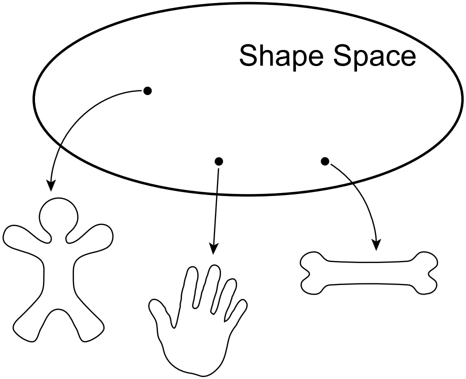

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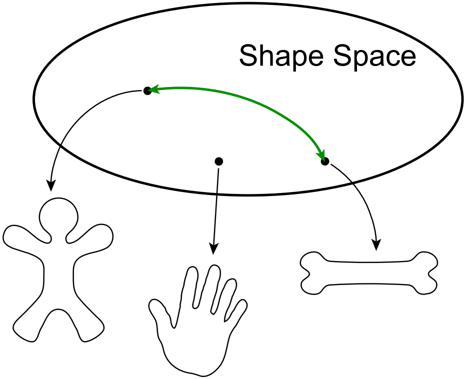
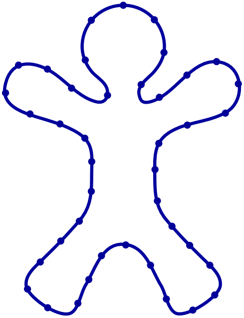


A metric space structure provides a comparison between two shapes.

Kendall's Shape Space

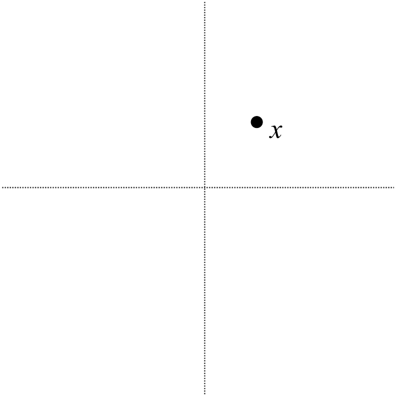


- Define object with k points.
- Represent as a vector in $\mathbb{R}^{2k}$.
- Remove translation, rotation, and scale.
- End up with complex projective space, $\mathbb{CP}^{k-2}$.

Kendall, 1984

Quotient Spaces

What do we get when we “remove” scaling from $\mathbb{R}^2$?



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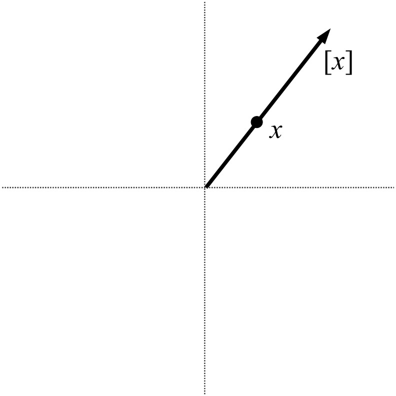

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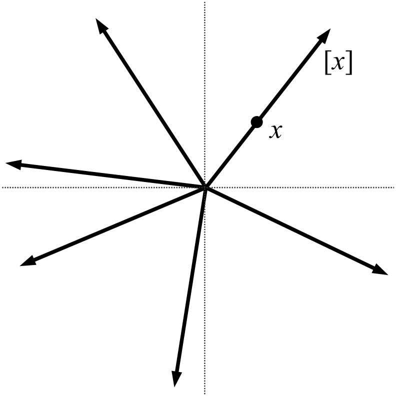

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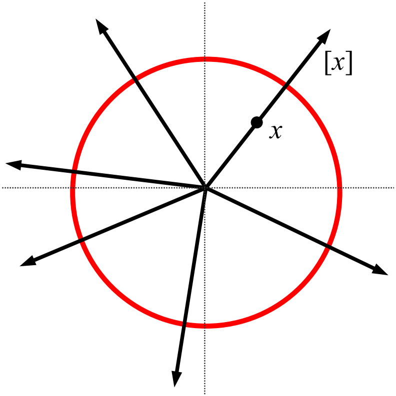

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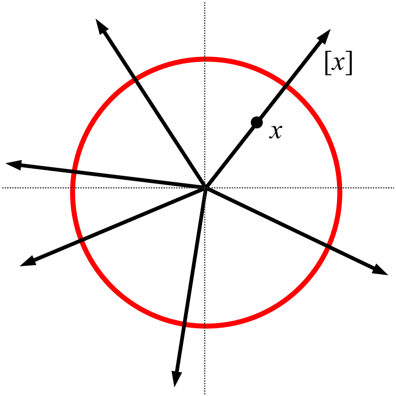

Notation: $[x] \in \mathbb{R}^2/\mathbb{R}^+$

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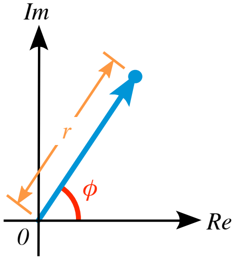
- An object is then a point $(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k$.
- Removing **translation** leaves us with $\mathbb{C}^{k-1}$.
- How to remove **scaling** and **rotation**?

Scaling and Rotation in the Complex Plane



Recall a complex number can be written as $z = re^{i\phi}$, with modulus r and argument ϕ.

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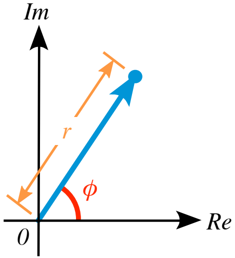

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Complex Multiplication:

$$se^{i\theta} * re^{i\phi} = (sr)e^{i(\theta+\phi)}$$

Multiplication by a complex number $se^{i\theta}$ is equivalent to scaling by s and rotation by θ.

Removing Scale and Translation

Multiplying a centered point set, $\mathbf{z} = (z_1, z_2, \ldots, z_{k-1})$, by a constant $w \in \mathbb{C}$, just rotates and scales it.

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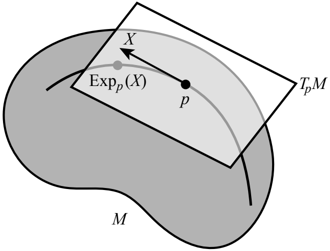
This gives complex projective space $\mathbb{CP}^{k-2}$ – much like the sphere comes from equivalence classes of scalar multiplication in $\mathbb{R}^n$.

The Exponential and Log Maps



- The exponential map takes tangent vectors to points along geodesics.
- The length of the tangent vector equals the length along the geodesic segment.
- Its inverse is the log map – it gives distance between points: $d(p, q) = \| \mathrm{Log}_p(q) \|$.

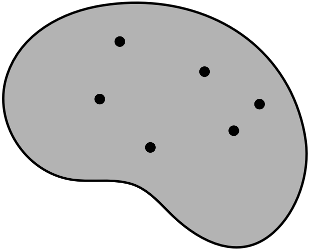
Intrinsic Means (Fréchet, 1948)

The *intrinsic mean* of a collection of points $x_1, \dots, x_N$ on a metric space M is

$$\mu = \arg\min_{x \in M} \sum_{i=1}^{N} d(x, x_i)^2,$$

If M is a Riemannian manifold, d is geodesic distance.

Computing Means



Gradient Descent Algorithm:

Input: $\mathbf{x}_1, \ldots, \mathbf{x}_N \in M$

$\mu_0 = \mathbf{x}_1$

Repeat:

$$\delta\mu = \frac{1}{N} \sum_{i=1}^{N} \mathrm{Log}_{\mu_k}(\mathbf{x}_i)$$

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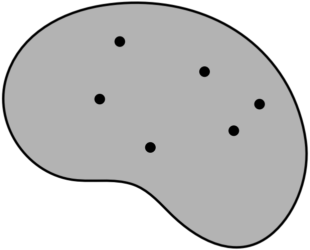

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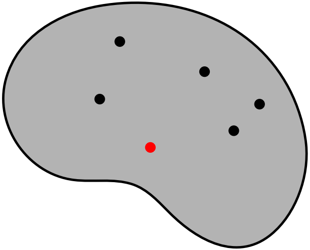

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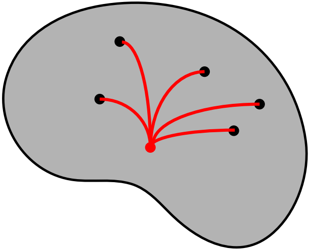

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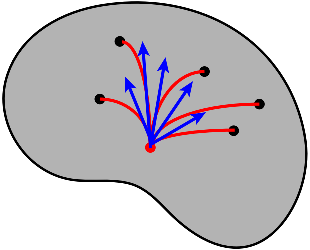

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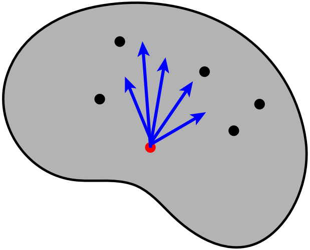

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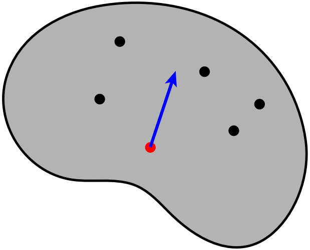

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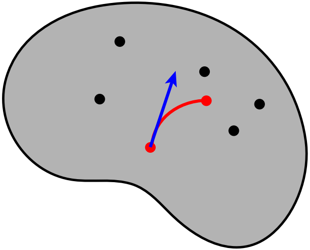

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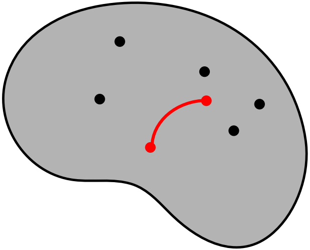

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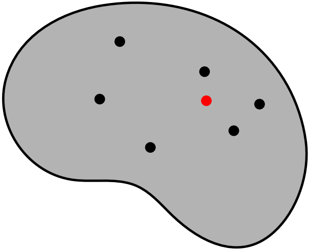

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Principal Geodesic Analysis

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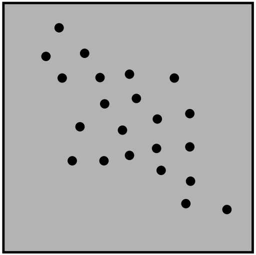

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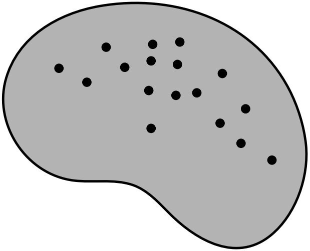

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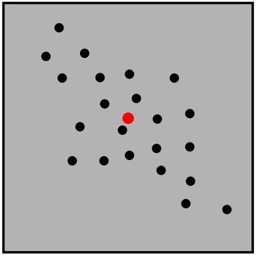

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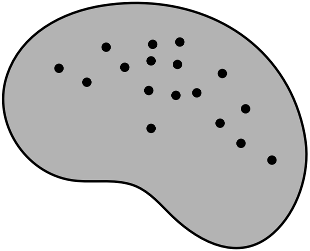

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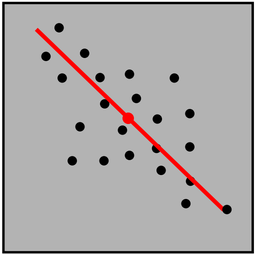

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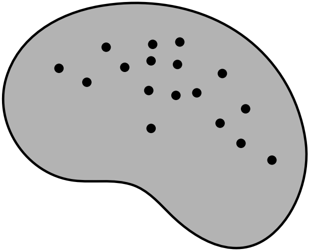

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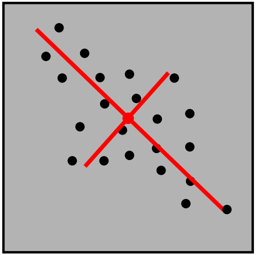

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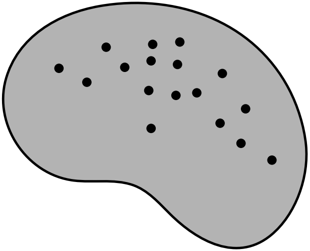

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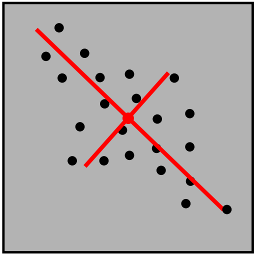

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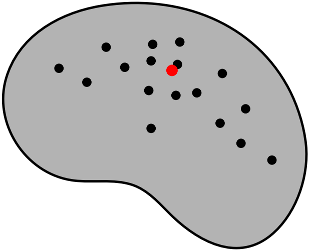

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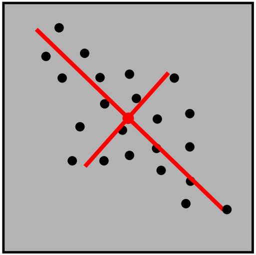

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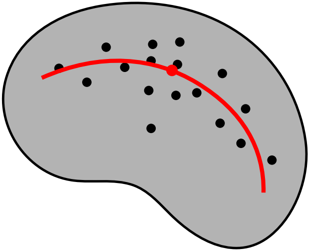

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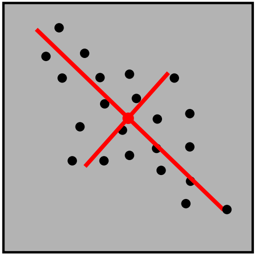

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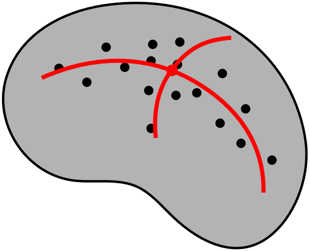
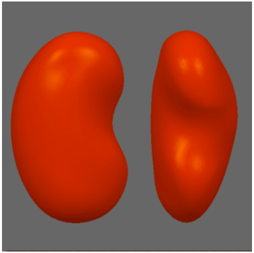
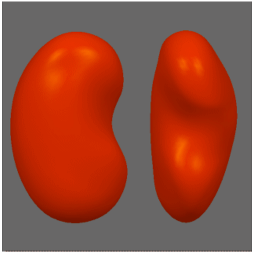


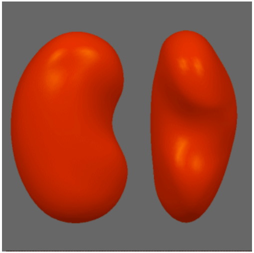
PGA of Kidney



Mode 1



Mode 2



Mode 3

Robust Statistics: Motivation

- The mean is overly influenced by outliers due to sum-of-squares.
- Robust statistical description of shape or other manifold data.
- Deal with outliers due to imaging noise or data corruption.
- Misdiagnosis, segmentation error, or outlier in a population study.

Mean vs. Median in $\mathbb{R}^n$

Mean: least-squares problem

$$\mu = \arg\min_{x \in \mathbb{R}^n} \sum \|x - x_i\|^2$$

Closed-form solution (arithmetic average)

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$$\mu = \arg\min_{x \in \mathbb{R}^n} \sum \|x - x_i\|^2$$

Closed-form solution (arithmetic average)

Geometric Median, or Fermat-Weber Point:

$$m = \arg\min_{x \in \mathbb{R}^n} \sum \|x - x_i\|$$

No closed-form solution

Weiszfeld Algorithm in $\mathbb{R}^n$

- Gradient descent on sum-of-distance:

$$m_{k+1} = m_k - \alpha G_k,$$

$$G_k = \sum_{i \in I_k} \frac{m_k - x_i}{\|x_i - m_k\|} \Bigg/ \left(\sum_{i \in I_k} \|x_i - m_k\|^{-1} \right)$$

- Step size: $0 < \alpha \leq 2$
- Exclude singular points: $I_k = \{i : m_k \neq x_i\}$
- Weiszfeld (1937), Ostresh (1978)

Geometric Median on a Manifold

The geometric median of data $x_i \in M$ is the point that minimizes the sum of geodesic distances:

$$m = \arg\min_{x \in M} \sum_{i=1}^{N} d(x, x_i)$$

Fletcher, et al. CVPR 2008 and NeuroImage 2009.

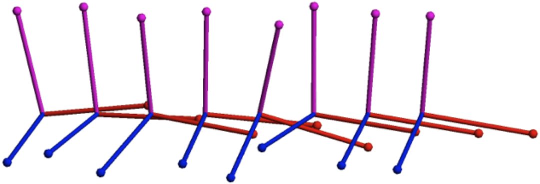
Weiszfeld Algorithm for Manifolds

Gradient descent:

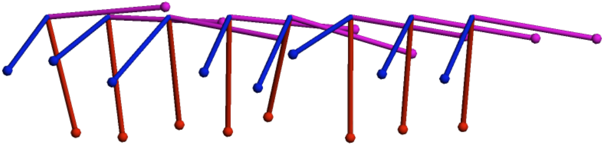
$$
\begin{aligned}
m_{k+1} &= \mathrm{Exp}_{m_k}(\alpha v_k), \\
v_k &= \sum_{i \in I_k} \frac{\mathrm{Log}_{m_k}(x_i)}{d(m_k, x_i)} \Bigg/ \left(\sum_{i \in I_k} d(m_k, x_i)^{-1} \right)
\end{aligned}
$$

Example: Rotations

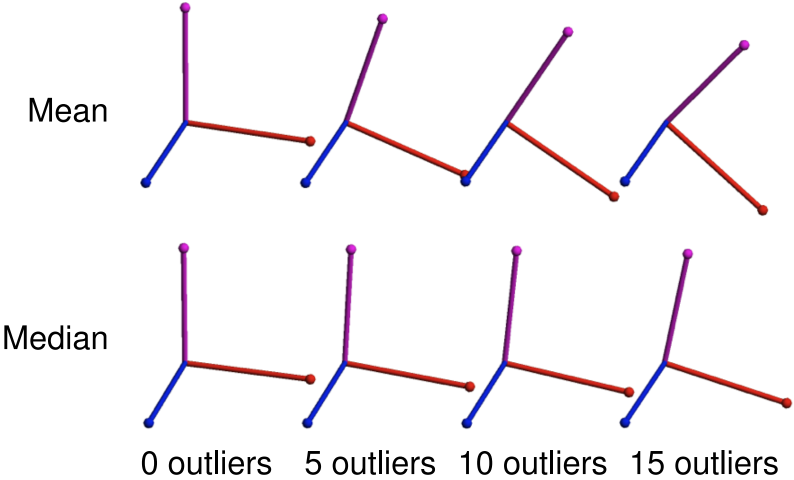
Input data: 20 random rotations



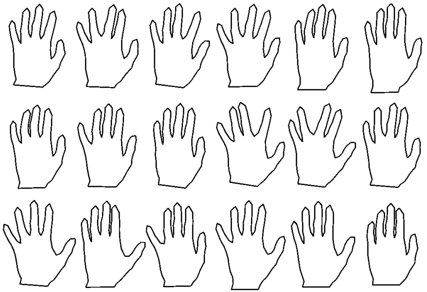
Outlier set: random, rotated 90°



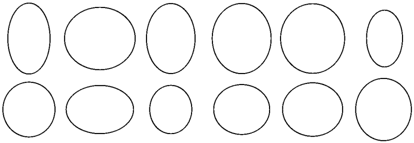
Example: Rotations



Example on Kendall Shape Spaces



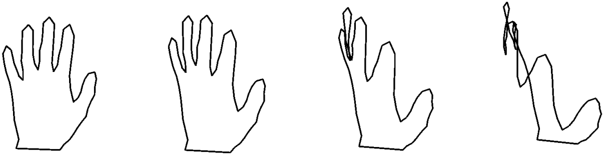
Hand shapes



Outliers

Example on Kendall Shape Spaces

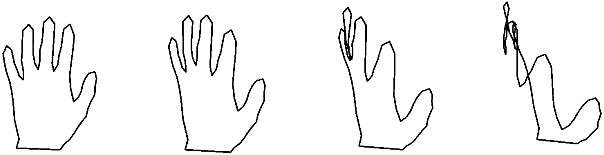
Mean:



Outliers: 0 2 6 12

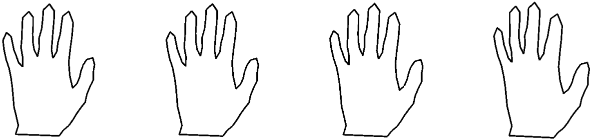
Example on Kendall Shape Spaces

Mean:



Outliers: 0 2 6 12

Median:



Outliers: 0 2 6 12

Image Metamorphosis

- Metric between images
- Includes both deformation and intensity change

$$U(v_t, I_t) = \int_0^1 \|v_t\|_V^2 \, dt \; + \; \frac{1}{\sigma^2} \int_0^1 \left\| \frac{dI_t}{dt} + \langle \nabla I_t, v_t \rangle \right\|_{L^2}^2 dt$$

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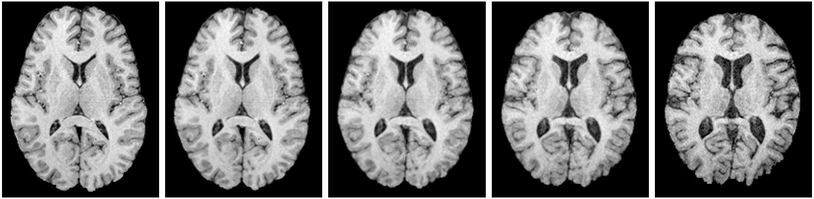
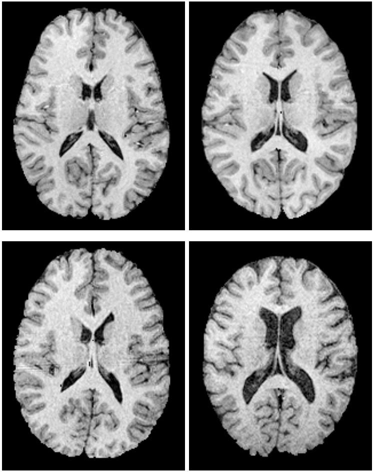
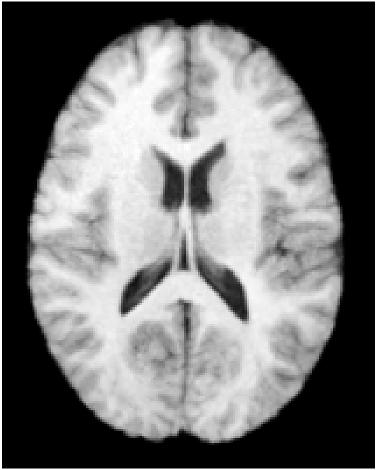


Example: Metamorphosis



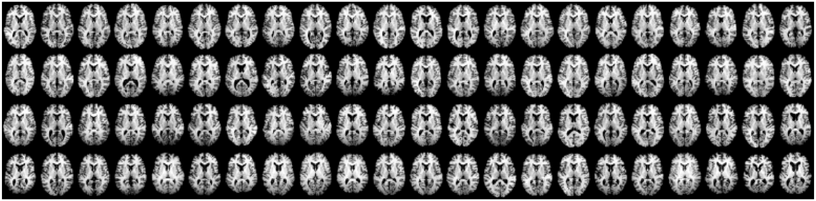
Input Data



Median Atlas

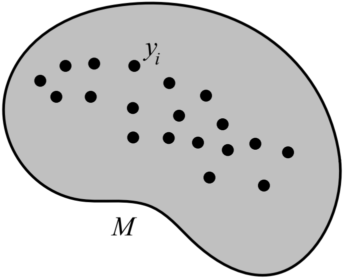
Describing Shape Change

- How does shape change over time?
- Changes due to growth, aging, disease, etc.
- Example: 100 healthy subjects, 20–80 yrs. old



- We need regression of shape!

Regression on Manifolds

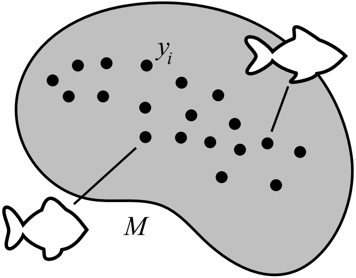


Given:

Manifold data: $y_i \in M$

Scalar data: $x_i \in \mathbb{R}$

Regression on Manifolds



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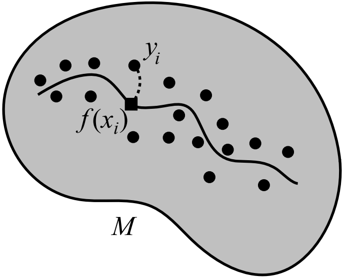
Regression on Manifolds



Given:

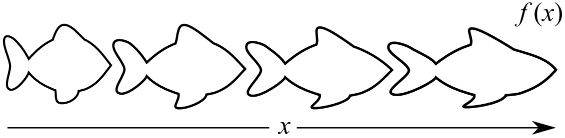
Manifold data: $y_i \in M$

Scalar data: $x_i \in \mathbb{R}$

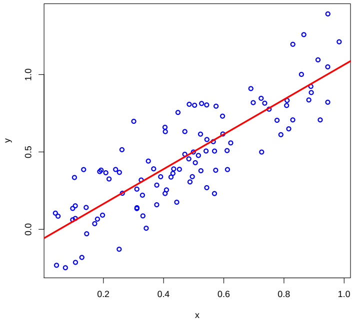
Want:

Relationship $f : \mathbb{R} \to M$

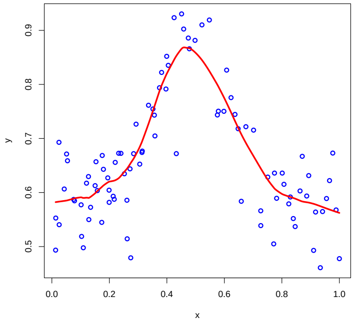
"how x explains y"



Parametric vs. Nonparametric Regression



Linear Regression



Kernel Regression

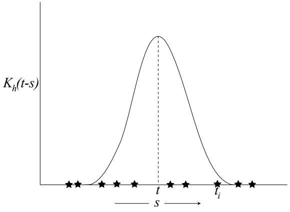
Kernel Regression (Nadaraya-Watson)

Define regression function through weighted averaging:

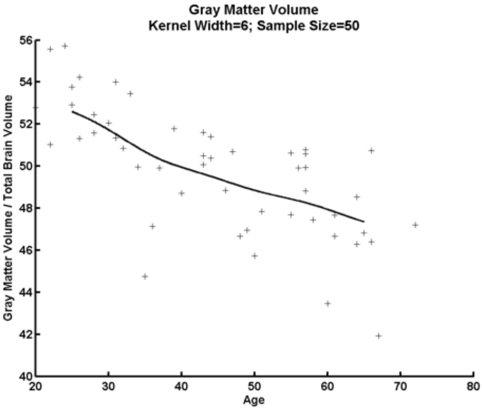
$$f(t) = \sum_{i=1}^{N} w_i(t) Y_i$$

$$w_i(t) = \frac{K_h(t - T_i)}{\sum_{i=1}^{N} K_h(t - T_i)}$$

Example: Gray Matter Volume

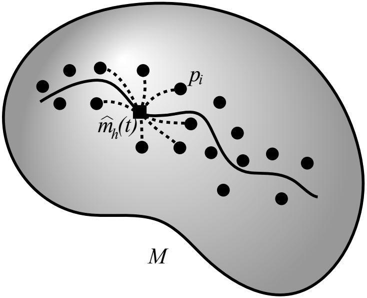


$$w_i(t) = \frac{K_h(t - T_i)}{\sum_{i=1}^{N} K_h(t - T_i)}$$



$$f(t) = \sum_{i=1}^{N} w_i(t) Y_i$$

Manifold Kernel Regression



Using Fréchet weighted average:

$$\hat{m}_h(t) = \arg\min_y \sum_{i=1}^{N} w_i(t) d(y, Y_i)^2$$

Davis, et al. ICCV 2007

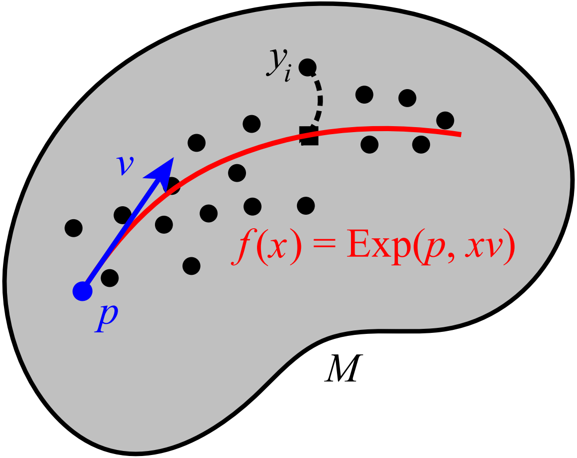
Geodesic Regression

- Generalization of linear regression.
- Find best fitting geodesic to the data (x_i, y_i).
- Least-squares problem:

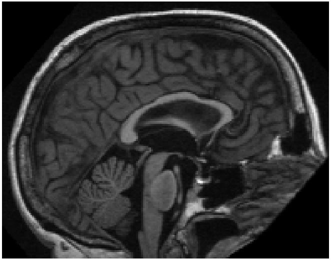
$$E(p, v) = \frac{1}{2} \sum_{i=1}^{N} d\left(\mathrm{Exp}(p, x_i\, v), y_i\right)^2$$

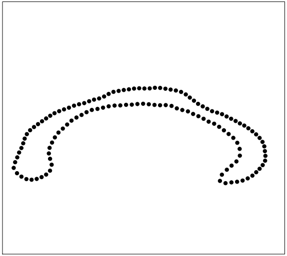
$$(\hat{p}, \hat{v}) = \arg \min_{(p,v) \in TM} E(p, v)$$

Geodesic Regression



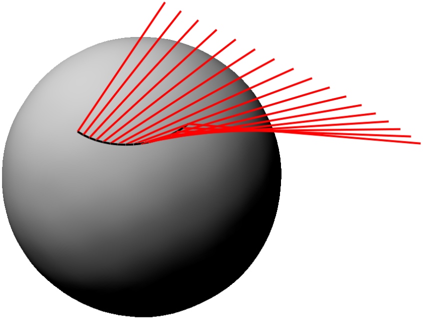
Experiment: Corpus Callosum





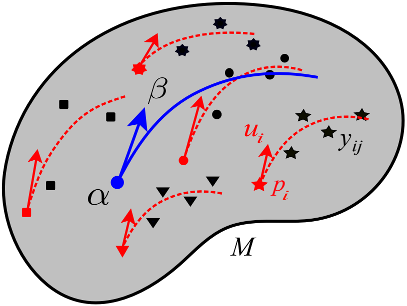
- The corpus callosum is the main interhemispheric white matter connection
- Known volume decrease with aging
- 32 corpus callosi segmented from OASIS MRI data
- Point correspondences generated using ShapeWorks `www.sci.utah.edu/software/`

The Tangent Bundle, TM



- Space of all tangent vectors (and their base points)
- Has a natural metric, called Sasaki metric
- Can compute geodesics, distances between tangent vectors

Longitudinal Models



Individual geodesic trends:

$$Y_i = \mathrm{Exp}(\mathrm{Exp}(p_i, X_i u_i), \epsilon_i)$$

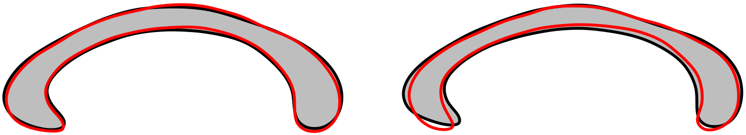
Average group trend (in TM):

$$(p_i, u_i) = \mathrm{Exp}_S((\alpha, \beta), (v_i, w_i))$$

Muralidharan & Fletcher, CVPR 2012

Longitudinal Corpus Callosum Experiment

- 12 subjects with dementia, 11 healthy controls
- 3 time points each, spanning 6 years



Healthy Controls   Dementia Patients

Statistically significant: $p = 0.027$

Open Problems

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- Estimator properties: consistency, efficiency

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- Estimator properties: consistency, efficiency
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- Clustering, classification
- Sparsity-like principles

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- Brad Davis (Kitware)

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